Computer Science 238: Optimized Democracy

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Abstract

These are notes[1] for Harvard's *Computer Science 238*, a graduate class on optimizing democracy, as taught by Professor Ariel Procaccia in Spring 2022.

Course description: The course examines the mathematical and algorithmic foundations of democracy, running the gamut from theory to applications. The goal is to provide students with a rigorous perspective on, and a technical toolbox for, the design of better democratic systems. Topics include computational social choice (identifying optimal voting rules), fair division with applications to political redistricting (avoiding gerrymandering) and apportionment (allocating seats on a representative body), sortition (randomly selecting citizens' assemblies), liquid democracy (transitively delegating votes), and weighted voting games (analyzing legislative power through cooperative game theory).

Contents

[1] With thanks to Eric K. Zhang for the template.

1 January 24th, 2022

1.1 Introduction

We begin with a quote from Pia Mancini: "Politics is solving today's problems with yesterday's tools."

The objective of this course is to provide students with a rigorous perspective and a technical toolbox for the design of better democratic systems.

From an undergraduate perspective, we seek to understand existing ideas and apply them. From a graduate perspective, creativity is required to generate new ideas.

1.2 Logistics

This course is split into two components: voting and allocation.

We then reviewed the syllabus. Topics covered will be

Voting	Allocation
Voting rules	Cake cutting
The axiomatic approach	Rent division
Strategic manipulation	Indivisible goods
Restricted preferences	Random assignment
Electoral competition	Sortition
The epistemic approach	Apportionment in the 19th century
Liquid democracy	Apportionment in the 20th century
Committee elections	Redistricting as cake cutting
Participatory budgeting	Identifying gerrymandered maps
	The electoral college

Table 1: Caption

The grade breakdown is 40% theoretical homework, 15% class participation, and 45% research project.

We will use Ed as a discussion forum. Course TFs are Daniel Halpern and Haneul Shin. Professor Procaccia's OH are by appointment.

All lectures will be recorded.

1.3 Voting rules

Definition 1.1 (Plurality)**.** In *plurality* voting, each person votes for a single alternative and the alternative with the most points win.

This is a high problematic voting rule.

There are many different ballot types: rankings, approvals, scores or stars. We will focus on rankings.

1.3.1 Borda

We introduce Jean-Charles de Borda, a mathematician, engineer, and naval officer. Also the instigator of the metric system.

Definition 1.2 (Borda count)**.** Each voter awards $m-k$ points to the alternative placed in the k-th position, where m is the number of alternatives.

Definition 1.3 (Single transferable vote)**.** In *single transferable vote*, votes are tabulated in rounds where in each round, the alternative with the lowest plurality score is eliminated. The last alternative remaining is the winner.

Example 1.4. STV is used in the USA for statewide elections in ME and AK. It is used in AUS for parliamentary elections, in IE for all public elections, and in Ontario, CAN for municipal elections.

1.3.2 Condorcet

We now introduce Marquis de Condorcet, a philosopher, mathematician and a nobleman. He is known for dying mysteriously in prison.

The *Condorcet paradox* is a paradox where the preferences of the majority may be cyclical and we cannot recover a unique winner to the election.

Definition 1.5 (Condorcet winner)**.** A *Condorcet winner* is an alternative that defeats every other alternative in a head-to-head comparison.

Definition 1.6 (Condorcet consistent)**.** A rule is *Condorcet consistent* if it always elects a Condorcet winner whenever it is presented with a profile that contains one.

1.3.3 Llull

Ramon Llull was a monk, missionary, and philosopher remembered for publishing a medieval parenting guide.

Definition 1.7 (Llull's rule)**.** In Llull's rule, each alternative receives one point for each head-to-head comparison it wins (as well as for tied comparisons).

Note. Llull's rule is Condorcet consistent.

1.3.4 Dodgson

Charles Lutwidge Dodgson was a professor of mathematics at Oxford.

Definition 1.8 (Dodgson score)**.** The *Dodgson score* of an alternative x is the minimum number of swaps between adjacent alternatives needed to make x a Condorcet winner. The winner is the alternative with minimum score.

Note. Dodgson's rule is Condorcet consistent. Moreover, it is NP-hard to compute.

Example 1.9. Consider the following example: We note that the different voting schemes produce different

33 voters	16 voters	3 voters	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	e
d	e	a	d	b	d
e	a	e	a	a	a

Table 2: Different voting mechanisms.

outcomes. Using plurality, a is the winner. Using the Borda count, b is the winner. The Condorcet winner is c and STV gives us d.

2 January 26th, 2022

2.1 The axiomatic approach

Consider the axioms of Euclidean geometry.

Social choice theory similarly tries to analyze group decisions making through an axiomatic lens. We note taht some axioms are much more intuitive than others.

2.2 The voting model

We consider the following voter model:

- A set of *voters* $N = \{1, \cdots, n\}$
- A set of *alternatives* $A, |A| = m$
- Each voter has a *ranking* $\sigma_i \in \mathcal{L}$ over alternatives, $x \succ_{\sigma_i} y$ means that voter i prefers x to y
- A *preference profile* $\boldsymbol{\sigma} \in \mathcal{L}^n$ is a collection of all voters' rankings
- A *social choice function* is a function $f : \mathcal{L}^n \to A$ and a *social welfare function* is $f : \mathcal{L}^n \to \mathcal{L}$

2.2.1 Two alternatives

If there are two alternatives, the social choice and social welfare functions coincide, so let's use social choice functions.

In this case, the *majority* seems to be the only sensible rule.

There are three axioms satisfied by the majority:

1. **Anonymity.** The rule is indifferent to voters' identities. Permuting the assignment of voters to rankings does **not** change the outcome.
2. **Neutrality.** The rule is indifferent to alternatives' identities. Permuting the alternatives permutes the outcome in the same way.
3. **Monotonicity.** Pushing x upwards in the votes doesn't harm x. For $m = 2$, flipping voters from $y \to x$ cannot flip the outcome from $x \to y$.

Theorem 2.1 (May's theorem)**.** *Assume $m = 2$ and n odd. Then $f : \mathcal{L}^n \to A$ is anonymous, neutral and monotonic $\iff$ it is majority.*

The case of n even requires more care for ties, but it is essentially the same.

Proof. AFTSOC $\exists$ profile $\boldsymbol{\sigma} : b$ is selected with $t < n/2$ votes. Obtain $\boldsymbol{\sigma'}$ by letting all voters flip their votes. By neutrality, $f(\boldsymbol{\sigma'}) = a$. Flip b votes in $\boldsymbol{\sigma'}$ to obtain $\boldsymbol{\sigma''}$ where a has $n - t$ votes. By monotonicity, $f(\boldsymbol{\sigma''}) = a$. But by anonymity, it holds that $f(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma''})$, which is a contradiction. □

2.3 The general case

We want to design a great social welfare function for $m > 2$. We know that the majority is only reasonable to aggregated preferences over pairs of alternatives.

2.3.1 Arrow

Kenneth Arrow was a professor at Harvard and Stanford and the 1972 Nobel laureate in economics.

Arrow's axioms are as follows:

1. **Unanimity.** If all voters rank x above y, the so does the social welfare function.
2. **Independence of irrelevant alternatives.** The social ranking over x and y only depends on each voter's ranking restricted to x and y.
3. **Nondictatorship.** There is no voter that can unilaterally determine the social ranking.

Theorem 2.2 (Arrow)**.** *Assume that $m \geq 3$. Then there does not exist $f : \mathcal{L}^n \to \mathcal{L}$ that satisfies unanimity, IIA, and nondictatorship.*

Dictatorship satisfies unanimity and IIA, so the theorem is a characterization of dictatorship.

Proof. Let $b \in A$. We will first show that if there exists a profile $\boldsymbol{\sigma}$ is such that b is at the top or bottom of each σ_i, then b is at the top or bottom of $f(\boldsymbol{\sigma})$. Suppose not and $\exists a, b : a \succ_{f(\boldsymbol{\sigma})} b \succ_{f(\boldsymbol{\sigma})} c$. By IIA, if $\boldsymbol{\sigma'}$ is obtained by every voter moving c above a, then it still holds that $a \succ_{f(\boldsymbol{\sigma'})} b$ and $b \succ_{f(\boldsymbol{\sigma'})} c \implies a \succ_{f(\boldsymbol{\sigma'})} c$. This is a contradiction to the unanimity at $\boldsymbol{\sigma'}$.

We will now show that there is a voter i^* that can move b from the bottom to the top of the social ranking by changing their vote in a profile. We can define profiles $\boldsymbol{\sigma}^0, \cdots, \boldsymbol{\sigma}^n$ where all voters rank b last in $\boldsymbol{\sigma}^0$ and $\boldsymbol{\sigma}^i$ is obtained from $\boldsymbol{\sigma}^{i-1}$ by i pushing b to the top. By unanimity, b is at the bottom of $f(\boldsymbol{\sigma}^0)$ and at the top of $f(\boldsymbol{\sigma}^n)$. The position of b first changes in $f(\boldsymbol{\sigma}^{i^*})$ and by step 1, it must change from top to bottom.

i^* is a dictator over any pair $\{a, c\}$ not involving b. WLOG we can show that i^* can force $a \succ c$. We can obtain $\boldsymbol{\pi}$ from $\boldsymbol{\sigma}^{i^*}$ by letting i^* rank $a \succ_{\pi_{i^*}} b \succ_{\pi_{i^*}} c$ and letting others arbitrarily rank a and c while keeping the position of b. The order of $\{a, b\}$ is the same in $\boldsymbol{\sigma}^{i^*-1}$ and $\boldsymbol{\pi} \implies a \succ_{f(\boldsymbol{\pi})} b$ by IIA. The order of $\{b, c\}$ is the same in $\boldsymbol{\sigma}^{i^*}$ and $\boldsymbol{\pi} \implies b \succ_{f(\boldsymbol{\pi})} c$ by IIA $\implies a \succ_{f(\boldsymbol{\pi})} c$ by IIA. All rankings of $\{a, c\}$ are arbitrary except that of i^*.

Hence. i^* is a dictator. By step 3, there is i^{**} that is a dictator for every pair not involving $c \neq b$ like $\{a, b\}$. But we know that i^* can affect the social ordering of $\{a, b\}$ by moving from $\boldsymbol{\sigma}^{i^*-1} \to \boldsymbol{\sigma}^{i^*} \implies i^* = i^{**} \implies i^*$ dictates the social order on every pair except $\{b, c\}$. $\square$

Note. Another application of this argument to a dictator for every pair not involving $a \in \{b, c\}$ completes the proof.

Note. Monotonicity is natural for two alternatives, but not so obvious for more. STV is **not** monotonic.

3 January 31st, 2022

Announcements

- The audio in previous lectures is not so good
- Participation is important in this class!

3.1 Strategic manipulation

Recall the voting model. We have:

- A set of *voters* $N = \{1, \cdots, n\}$
- A set of *alternatives* $A, |A| = m$
- Each voter has a *ranking* $\sigma_i \in \mathcal{L}$ over alternatives, $x \succ_{\sigma_i} y$ means that voter i prefers x to y
- A *preference profile* $\boldsymbol{\sigma} \in \mathcal{L}^n$ is a collection of all voters' rankings
- A *social choice function* is a function $f : \mathcal{L}^n \to A$ and a *social welfare function* is $f : \mathcal{L}^n \to \mathcal{L}$

So far, our voters were honest. We will now consider a case of strategic manipulation.

Example 3.1 (Strategic manipulation)**.** We can consider a set of four voters using the Borda count. In the top profile, b wins and in the bottom profile, a wins.

3.1.1 Strategy-proofness

Note. We will let $\boldsymbol{\sigma}_{-i} \equiv (\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_n)$.

Definition 3.2 (Strategy-proofness)**.** A social choice function f is *strategy proof* (SP) if a voter can never benefit from lying about their preferences. That is: $\forall \sigma \in \mathcal{L}^n, \forall i \in N, \forall \sigma_i' \in \mathcal{L}$, we have

$$f(\boldsymbol{\sigma}) \succeq_{\sigma_i} f(\sigma_i', \boldsymbol{\sigma}_{-i}). \tag{1}$$

Note (Plurality)**.** The maximum number of alternatives for which plurality is SP is $m = 2$.

For $m = 3$, we note that if you know your most preferred candidate cannot win, you can simply deviate and vote for your second most-preferred to prevent your least preferred candidate from winning.

Theorem 3.3 (Gibbard-Satterthwaite impossibility)**.** *Let $m \geq 3$. Then a social choice function f is SP and onto A (any alternative can win) $\iff$ f is dictatorial.*

In other words, any voting rule that is onto and nondictatorial is manipulable.

G-S proof sketch. We will provide a sketch of G-S. Consider the following lemmas:

Lemma 3.4 (Strong monotonicity)**.** *If f is an SP function, $\boldsymbol{\sigma}$ a profile, $f(\boldsymbol{\sigma}) = a \implies f(\boldsymbol{\sigma'}) = a$ for all profiles $\boldsymbol{\sigma'} : \forall x \in A, i \in N$:*

$$a \succ_{\sigma_i} x \implies a \succ_{\sigma_i'} x. \tag{2}$$

Lemma 3.5 (Unanimity)**.** *If f is an SP and onto function, $\boldsymbol{\sigma}$ a profile, then*

$$\forall i \in N, a \succ_{\sigma_i} b \implies f(\boldsymbol{\sigma}) \neq b. \tag{3}$$

We will prove these lemmas in the homework. Let us assume that $m \geq n$ and *neutrality*:

$$f(\pi(\boldsymbol{\sigma})) = \pi(f(\boldsymbol{\sigma})), \qquad \forall \pi : A \to A \tag{4}$$

1	2	3	4
a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c
e	e	e	e

Table 3: $\boldsymbol{\sigma}$

1	2	3	4
a	d	d	d
b	a	a	a
c	b	b	b
d	c	c	c
e	e	e	e

1	2	3	4
a	d	d	d
b	b	a	a
c	c	b	b
d	e	c	c
e	a	e	e

1	2	3	4
a	d	d	d
b	b	b	a
c	c	c	b
d	e	e	c
e	a	a	e

1	2	3	4
a	d	d	d
b	b	b	b
c	c	c	c
d	e	e	e
e	a	a	a

Figure 1

We will consider an example $n = 4, A = \{a, b, c, d, e\}$ and the following profiles We will suppose that $f(\boldsymbol{\sigma}) = a$ because this is symmetric with respect to alternatives $\{b, c, d\}$. Unanimity guarantees that $f(\boldsymbol{\sigma}) \neq e$.

We can rotate a to the bottom to create new profiles.

Unanimity implies that $f(\boldsymbol{\sigma}^j) \notin \{b, c, e\}$ and strategy-proofness $\implies f(\boldsymbol{\sigma}^j) \neq d \implies f(\boldsymbol{\sigma}^j) = a$.

We note that by strong monotonocity, $f(\boldsymbol{\sigma}) = a \forall \boldsymbol{\sigma}$ where 1 ranks a first. By neutrality, we note that voter 1 is a dictator for any alternative. □

For $m \geq 3$, all common voting rules are onto and nondictatorial. We can restrict our attention to two alternatives and this will force the voting rule to become strategy-proof and nondictatorial.

3.2 Computational problem

The central question is how to design a *reasonable* voting rule where manipulation is computationally hard.

3.2.1 f-Manipulaton problem

Given votes of non-manipulators and a preferred alternative p, can a manipulator cast a vote that makes p *uniquely* win under f?

Example 3.6. Under Borda, $p = a$ is possible.

A greedy algorithm is as follows: rank p in first place. While there are unranked alternatives, if there is an alternative that can be placed in the next spot without preventing p from winning, place this alternative. Else, return *false*.

Example 3.7 (Borda)**.** We went over the example of Borda using the greedy algorithm above.

Example 3.8 (Llull)**.** We went over the example of Llull using the greedy algorithm above.

The following theorem gives a sufficient condition for the greedy algorithm to work.

Theorem 3.9. *Fix $i \in N$ and the votes of other voters. Let f be a rule* $: \exists s(\sigma_i, x)$*;*

1. *For all σ_i, f chooses an alternative that uniquely maximizes $s(\sigma_i, x)$*
2. *If $\{y : y \prec_{\sigma_i} x\} \subseteq \{y : y \prec_{\sigma'_i} x\} \implies s(\sigma_i, x) \leq s(\sigma'_i, x)$.*

Then the greedy algorithm decides the f-Manipulation problem correctly.

Proof. Suppose the greedy algorithm failed, producing a partial ranking σ_i. AFTSCOC σ_i' makes p win. Let U be the alternatives not ranked in σ_i and u the highest ranked alternative ranked in U according to σ_i'. Complete σ_i by adding u first, then others arbitrarily.

By property 2 we have $s(\sigma_i, p) \geq s(\sigma_i', p)$. Property 1 and σ_i' makes p the winner $\implies$ $s(\sigma_i', p) > s(\sigma_i', u)$. Property 2 gives us $s)\sigma_i', u) \geq s(\sigma_i, u)$. Then observe that $s(\sigma_i, p) > s(\sigma_i, u)$ so the algorithm should not have failed and could have inserted u next, a contradiction. □

Note (Hard to manipulate rules)**.** Two NP-hard to manipulate rules have been found: single transferable vote (SVT) and Llull with tie-breaking, but these are worst-case hard. This isn't necessarily an obstacle to manipulation in the average case.

There is still not an answer to if such a rule exists.

4 February 2nd, 2022

4.1 Restricted preferences

The Gibbard-Satterthwaite theorem requires a full preference domain where each ranking of alternatives is possible. We can circumvent this theorem if we restrict the preferences.

We assume an ordering $\preceq$ over the set of alternatives A.

Definition 4.1 (Single-peaked preferences)**.** Voter i has *single-peaked preferences* if there is a *peak* $x^* \in A$: $y < z \le x^* \implies z \succ_{\sigma_i} y$ and $y > z \ge x^* \implies z \succ_{\sigma_i} y$.

Graphically, we can think about the preferences having a global maximum over all alternatives.

Proposition 4.2. *Given an odd number of voters with single-peaked preferences, a Condorcet winner exists and is given by the median peak.*

Intuitively, this is because a majority of voters will prefer the median to any alternative to its right or left, depending on where the voter is relative to the median.

Proposition 4.3. *If we have single-peaked preferences, then the voting rule that selects the median peak is* ***strategy-proof****.*

Proof. Consider a voter whose true preference is left of the median. Then misreportined a peak on the same side of the median makes no difference. Reporting another peak on the other side of the median is strictly worse. □

Corollary 4.3.1. *An extension of this is that any voting rule that selects the k-th order statistic is strategy proof.*

Proof. The same argument from above applies to the k-th order statistic. □

4.1.1 Strategy-proof rules

For single peaked preferences σ_i, we denote the peak as $P(\sigma_i)$.

Theorem 4.4 (Moulin 1980)**.** *An anonymous voting rule on single-peaked preferences is strategy-proof $\iff$ $\exists p_1, \cdots, p_{n+1} \in A$ called phantoms such that for all profiles $\boldsymbol{\sigma}$:*

$$f(\boldsymbol{\sigma}) = \text{med}(p_1, \cdots, p_{n+1}, P(\sigma_1), \cdots, P(\sigma_n)) \tag{5}$$

Example 4.5 (Phantoms)**.** For the median, consider odd n and placing $\frac{1}{2}(n+1)$ phantoms each at a_1 and a_m.

For the second order statistic, consider $n-1$ phantoms at a_1 and two at a_m.

If we want a constant function, $f \equiv x$, place $n+1$ phantoms at x. Note that this is also strategy-proof.

4.1.2 Facility location

We assume the following. Each *player* $i \in N$ has a *location* $x_i \in \mathbb{R}$. Given $\boldsymbol{x} = (x_1, \cdots, x_n)$, choose an alternative, a *facility location* $f(\boldsymbol{x}) = y \in \mathbb{R}$.

We will assume the following utility or cost function for each player:

$$\text{cost}(y, x_i) = |y - x_i|. \tag{6}$$

This defines very specific single-peaked preferences over the set of alternatives $\mathbb{R}$ where the peak of player i is x_i.

We can consider two objective functions: social cost — or social welfare — and maximum cost, given respectively by

$$\mathrm{sc}(y, \boldsymbol{x}) = \sum_i |y - x_i|, \qquad \mathrm{mc}(y, \boldsymbol{x}) = \max_{i \in N} |y - x_i|. \tag{7}$$

For the maximum cost, we want to make the most unhappy voter the happiest as possible.

Claim. For the social cost objective, the median is **optimal and strategy-proof**.

Claim. The optimal solution for the max cost objective is **not strategy-proof**.

This is because there is a misreport where we can move the maximum cost to the true preference.

4.2 Deterministic rules for maximum cost

We will thus think about a *relaxation* of the problem on our condition of strategy-proofness. We will thus consider approximations to the maximum cost.

Definition 4.6 (α-approximation)**.** A deterministic rule f is an α-approximation to the max cost if $\forall \boldsymbol{x} \in \mathbb{R}^n$:

$$\mathrm{mc}(f(\boldsymbol{x}), \boldsymbol{x}) \leq \alpha \min_{y \in \mathbb{R}} \mathrm{mc}(y, \boldsymbol{x}). \tag{8}$$

Note. The approximation ratio of the median to the max cost is 2. The median will always be between the leftmost and rightmost solutions. At optimal, it is half-way in between.

Theorem 4.7. *No deterministic strategy-proof rule has an approximation ratio* < 2 *to the max cost.*

By illustration. AFTSOC that we get a < 2 approximation. Consider two voters at locations x_1, x_2. For a < 2 approximation, the facility must be $y \in (x_1, x_2)$. Player 2 can misreport their location to move the facility location to the true preference. □

4.3 Randomized rules for maximum cost

Theorem 4.8. *We say that a randomized rule f gives an α-approximation to the max cost if* $\forall x \in \mathbb{R}^n$*:*

$$E[\mathrm{mc}(f(\boldsymbol{x}), \boldsymbol{x})] \leq \alpha \min_{y \in \mathbb{R}} \mathrm{mc}(y, \boldsymbol{x}). \tag{9}$$

Definition 4.9 (Left-right-middle (LRM) rule)**.** We choose the min x_i and max x_i each with probability $\frac{1}{4}$ and choose the average with probability $\frac{1}{2}$. The approximation ratio is

$$\frac{1}{2} \times 2\mathrm{opt} + \frac{1}{2}\mathrm{opt} = \frac{3}{2}. \tag{10}$$

Theorem 4.10. *LRM is strategy-proof in expectation.*

Proof. Consider a misreport by leftmost voter moving left by 2δ. The middle would move by δ at most. Then note that this voter (on average) loses $\frac{1}{4} \times 2\delta$ and (on average) gains $\frac{1}{2}\delta$. The sum of these is 0, there is no exppected gain with this deviation. □

Theorem 4.11. *No randomized strategy-proof mechanism has an approximation ratio* $< 3/2$.

Proof. Let $x_1 = 0, x_2 = 1, f(\boldsymbol{x}) = P$. We know that $\text{cost}(P, x_1) + \text{cost}(P, x_2) \geq 1$. WLOG, we can assume that $\text{cost}(P, x_2) \geq 1/2$.

Consider a manipulation by voter 2, given by $x_1 = 0, \hat{x}_2 = 2$. By strategy-proofness, the expected distance from $x_2 = 1$ is at least $1/2$.

Then the expected maximum cost is at least $3/2$ because $\forall y \in \mathbb{R}$, the maximum cost is $|y - 1| + 1$. □

5 February 7th, 2022

5.1 Electoral competition

We will model competition with some game theory.

Definition 5.1 (Normal-form game)**.** A game in *normal form* consists of a set of players N, a strategy set S and for each $i \in N$, the utility function $u_i : S^n \to \mathbb{R}$ which gives the utility of player i $u_i(s_1, \cdots, s_n)$ when each $j \in N$ plays strategy s_j.

Example 5.2 (Prisoner's dilemma)**.** We then consider the example of the prisoner's dilemma, where the Nash equilibrium (NE) is given by (D, D), both players defect.

Definition 5.3 (Nash equilibrium)**.** A *Nash equilibrium* is a vector of strategies $\boldsymbol{s} = (s_1, \cdots, s_n) \in S^n$: $\forall i \in N, s_i' \in S$, we have

$$u_i(\boldsymbol{s}) \geq u_i(s_i', \boldsymbol{s}_{-i}). \tag{11}$$

Example 5.4. Consider a two-player game with strategies $\{2, \cdots, 100\}$. If both players choose the same number, they get that number. If one player chooses s and the other t and $s < t$, the former player will get $s + 2$ and the latter will get $s - 2$.

The only NE in this game is $(2, 2)$. This is because we can remove strategies by dominating other strategies.

This example shows us that a NE will sometimes not predict the best outcome.

5.2 The Hotelling model

The Hotelling model is due to Harold Hotelling.

We model the political spectrum as the real line $\mathbb{R}$. There is a nonatomic distribution of voters, each with a peak in $\mathbb{R}$. Players are candidates who strategically choose positions $x_1, \cdots, x_n$. Each candidate attracts the votes of voters who are closet to them, which votes split equally in case of a tie.

In this model, two candidates seek to win a plurality of votes. The utility of each candidate is 1 if they win, 1/2 if they tie, and 0 if they lose. We can denote the median peak by m and assume for simplicity that it is unique.

5.2.1 Nash equilibrium for two candidates

If $x_2 < m$ the best response for 1 is all $x_1 : x_1 > x_2, \frac{1}{2}(x_1 + x_2) < m$. By symmetry, this is also true if $x_2 > m$. Note that $x_1 = m \iff x_2 = m \implies (m, m)$ is a Nash equilibrium. We can let the best strategy for player 1 be given by

$$B_1(x_2) = \begin{cases} \{x_1 : x_2 < x_1 < 2m - x_2\}, & x_2 < m \\ \{m\}, & x_2 = m \\ \{x_1 : 2m - x_2 < x_1 < x_2, & x_2 > m \end{cases}. \tag{12}$$

5.2.2 Policy-motivated candidates

We now consider candidates who care about policy, in addition to winning. Suppose i has a preferred position x_i^* and the utility depends on the distance between x_i^* and the position of the winner. If there is a tie, then candidates evaluate the induced lottery over winning positions.

Theorem 5.5. $x_1^* < m < x_2^* \implies (m, m)$ *is the unique Nash equilibrium.*

Sketch. We can rule out cases for which $(x_1, x_2) \neq (m, m)$. ☐

5.2.3 Introducing uncertainty

Consider a modification to the model: both candidates believe that the median peak m is distributed according to a distribution μ with *strictly positive density* over an interval I.

For $x_1 < x_2$, the probability that 1 wins is

$$\pi_1(x_1, x_2) = P_{m \sim \mu}\left[m < \frac{x_1 + x_2}{2}\right]. \tag{13}$$

Candidate i's utility for (x_1, x_2) is given by

$$\pi_1(x_1, x_2)U_i(x_1) + \pi_2(x_1, x_2)U_i(x_2) \tag{14}$$

where U_i is maximized at x_i^*.

Theorem 5.6. *If $x_1^*, x_2^* \in I$ and $x_1^* \neq x_2^*$, then in any Nash equilibrium (x_1, x_2), it holds that $x_1 \neq x_2$.*

Proof. If $x_1 = x_2 = x^*$ then x^* is enacted with probability 1. WLOG $x^* < x_1^*$. IF 1 moves to $x_1' \in (x^*, x_1^*)$ then $\pi(x_1', x_2) > 0$ and they are better off. □

6 February 9th, 2022

6.1 The epistemic approach

For Condorcet, voting is a collective quest for the truth. For him, enlightened voters try to judge which alternative best serves society.

Theorem 6.1 (Condorcet jury theorem (1785))**.** *Suppose there is one correct alternative and an incorrect alternative. There are n voters, each of which vote independently for the correct alternative with probability $p > 1/2$. Then the probability that the majority would be correct goes to 1 as $n \to \infty$.*

Proof. The modern proof follows directly from the weak law of large numbers. Consider

Lemma 6.2 (Weak law of large numbers)**.** *Let $X_1, X_2, \cdots$ be an infinite sequence of i.i.d. random variables with expectation μ. Then $\forall \epsilon > 0$:*

$$\lim_{n\to\infty} P\left[|\bar{X}_n - \mu| < \epsilon\right] = 1. \tag{15}$$

We simply take $\epsilon = p - 1/2$. □

Note. We assume a lot about how voters vote, and the probability of them voting "correctly" in each pairwise comparison.

6.1.1 Case of $m \geq 3$

In Condorcet's general model, there is a true ranking of alternatives. Each voter evaluates every pair independently and gets the comparison correct with probability $p > 1/2$.

The results are tallied in a *voting matrix*. We then find the *most probable* ranking by taking the majority opinion for each comparison; if a cycle forms, we successively delete the comparisons that have the least plurality.

Note. We note that there are cases where this process creates ambiguity and cycles. It is unclear if Condorcet meant to *reverse* the weakest edge.

6.1.2 Young's solution

Let M be the matrix of votes and π the true ranking. Maximum likelihood estimation (MLE) maximizes $P(M|\pi)$.

If we suppose the true ranking is $a \succ_\pi b \succ_\pi c$. The probability of observations $P(M|\pi)$ is given by

$$\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{11} p^{11}(1-p)^2. \tag{16}$$

This is the product of the possible combination of voters where $a \succ b, a \succ c, b \succ c$, respectively.

For $a \succ_\pi c \succ_\pi b, P(M|\pi)$ is given by

$$\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{2} p^2(1-p)^{11}. \tag{17}$$

If we take any ranking, the binomial coefficients are identical, but note that

$$P(M|\pi) \propto p^{\#\text{agree}}(1-p)^{\#\text{disagree}} \tag{18}$$

6.2 Relaxation

We want to relax the constraint where voters vote independently on pairwise alternatives. We introduce the *Kendall tau* distance.

Definition 6.3 (Kendall tau distance)**.** The *Kendall tau* distance between σ, σ' is defined as

$$d_{KT}(\sigma, \sigma') = |\{\{a, b\} : a \succ_\sigma b \land b \succ_{\sigma'} a\}| . \tag{19}$$

We can think about this as the *bubble sort distance.*

6.2.1 The Mallows model

The Mallows model is defined by a parameter $\phi \in (0, 1]$. The probability of a voter having the ranking σ given true ranking π is

$$P(\sigma|\pi) = \frac{\phi^{d_{KT}(\sigma,\pi)}}{\sum_\tau \phi^{d_{KT}(\tau,\sigma)}}. \tag{20}$$

The more disagreements the σ is closer to the ground truth π, the less likely it is. The distribution is relatively sharply concentrated near the ground truth ranking and exponentially decreases as we go away.

Note. This is the same as the Condorcet noise model where we let each voter vote for every pair. The process "restarts" if a cycle forms. We can convert the parameters and

$$\phi = \frac{1-p}{p}. \tag{21}$$

6.2.2 The Kemeny rule

Let us denote

$$Z_\phi \equiv \sum_\tau \phi^{d_{KT}(\tau,\pi)}. \tag{22}$$

Then the probability of observing a profile $\boldsymbol{\sigma}$ given true ranking π is

$$P(\boldsymbol{\sigma}|\pi) = \prod_{i\in N} \frac{\phi^{D_{KT}(\sigma_i,\pi)}}{Z_\phi} = \frac{\phi^{\sum_{i\in N} d_{KT}(\sigma_i,\pi)}}{(Z_\phi)^n}. \tag{23}$$

The MLE is the *Kemeny rule*: Given a preference profile $\boldsymbol{\sigma}$, return a ranking π that minimizes $\sum_{i\in N} d_{KT}(\sigma_i, \pi)$. We want to minimize the sum of disagreements between voters' profiles and the ground truth.

Note. This is equivalent to Young's solution. Young wanted to minimize the number of disagreements.

Theorem 6.4. *Computing the output of the Kemeny rule is NP-hard.*

Proof. We are going to reduce from the following problem.

Minimum Feedback Arc Set. Given a directed graph $G = (V, E), L \in \mathbb{N}$, is there $F \subseteq E : |F| \leq L$ and $(V, E \setminus F)$ is acyclic?

For each edge, we create a pair of voters that agree on the corresponding ordered pair of alternatives and disagree on everything else. For every pair and every ranking, it has to agree with one of these voters and disagree with the other — except on the pair of alternatives that induced the pair of voters.

Then $\exists$ an acyclic subgraph that deletes k edges $\iff$ $\exists$ a ranking that (beyond inevitable disagreements) disagrees with k pairs of voters. □

Note. There is a connection to Condorcet if we consider the *weighted* version of this problem. We seek to delete the edges of minimum weight greedily.

6.2.3 Kemeny in practice

In practice, Kemeny computation is formulated as an integer linear program. $\forall a, b \in A, x_{(a,b)} = 1 \iff a \succ b$ and $w_{(a,b)} = |\{i \in N : a \succ_{\sigma_i} b\}|$. We want to minimize

$$\sum_{(a,b)} x_{(a,b)} w_{(b,a)} \tag{24}$$

subject to

$$x_{(a,b)} + x_{(b,a)} = 1, \qquad x_{(a,b)} + x_{(b,c)} + x_{(c,a)} \leq 2, \qquad x_{(a,b)} \in \{0,1\} \tag{25}$$

$\forall a \neq b \neq c \in A$.

Note. Kemeney satisfies both Condorcet consistency and unanimity!

7 February 14th, 2022

Announcements

- Manon Revel is conducting a study on liquid democracy! Survey to be sent later today

Today, we will discuss a paradigm of democraccy called *liquid democracy*.

We recall the following forms of democracy. Note that ancient Athens used *direct democracy*, where votes went straight to candidates. The modern UK and United States is a *representative democracy* where candidates elect representatives that serve their interests in Parliament/government.

7.1 Liquid democracy

A *liquid democracy* tries to get the best of both worlds. Voters can vote directly on certain issues or delegate their vote to *another voter*.

If a delegates to b and b delegates to c, c will vote on behalf of all of a, b, c.

This deals with problems in a representative and direct democracy, namely representation and scaling issues.

Example 7.1 (Liquid democracy systems)**.** We can consider LiquidFeedback in Germany, DemocracyOS in Argentina, and Flux in Australia.

We will discuss both an objective epistemic model of liquid democracy and a more optimistic subjective model.

7.2 The epistemic model

We can consider an underlying labeled directed graph $G = (V, E, \boldsymbol{p})$ on n vertices where V is the set of voters and $(i, j) \in E$ if i knows j. There are two alternatives, correct and incorrect. Decisions will be made on a majority vote.

Each voter i has a *competence level* p_i, which is their probability of voting correctly.

Definition 7.2 (Approval)**.** We say *i approves j* if $(i, j) \in E$ and $p_j > p_i + \alpha$. We note that α is a parameter of the model.

We denote i's approved neighbors by $A_G(i)$.

Note. We make this assumption about approval to model liquid democracy to strengthen the negative result, i.e. show that a negative result is better motivated.

We will compare liquid democracy to direct democracies.

Example 7.3. Consider a star with n vertices. The leaves have $p_i = 0.4$ and the center has $p_i = 0.8$, $\alpha < 0.4$.

In a direct democracy, by the Condorcet jury theorem, the probability that the majority is correct $\to 0$ as $n \to \infty$. Under a liquid democracy, all the leaves delegate to the center and the probability of correctness is 0.8.

If we change the leaves' competency to 0.6, however, as we add more and more leaves, the limit approaches 1 and direct democracy is better.

7.2.1 Delegation mechanisms

We seek to get around the issues above with smarter delegation.

Definition 7.4 (Delegation mechanism)**.** A *delegation mechanism* M observes G and the approval relation, and outputs for each $i \in V$, a probability distribution over $A_G(i) \cup \{i\}$ that represents the probability that i delegates their vote to each approved neighbors or votes directly.

We denote the probability that a delegation mechanism M makes a correct decision on G by $P_M(G)$.

We define $P_M(G)$ using the following process:

1. Apply M to G
2. Sample the probability distribution for each vertex to obtain an acyclic delegation graph, where each sink i of the delegation graph has weight equal to the number of vertices with directed paths to i, including i. We know this is acyclic because voters can only approve neighbors who are strictly more competent.
3. Each sink i votes for the correct alternative with probability p_i
4. A decision is made based on weighted majority

7.2.2 Local delegation mechanisms

In everyday situations, there is no global mechanism. Instead, every voter independently decides who to delegate to, using only local information.

Definition 7.5 (Local delegation mechanism)**.** A *local delegation mechanism* is a mechanism where the distribution of each vertex i depends only on $\{j \in V : (i, j) \in E\}$ and $A_G(i)$.

We want to capture the idea that liquid democracy will do better than democracy under local delegation mechanisms. We will use a property called *do no harm.* We will define the following:

Definition 7.6 (Gain)**.** *Gain* is defined by

$$\text{gain}(M, G) = P_M(G) - P_D(G) \tag{26}$$

where D is direct voting.

Definition 7.7 (Do no harm (DNH))**.** Mechanism M satisfies the *do no harm* propery if $\forall \epsilon > 0 \exists n_0 \in \mathbb{N}$: $\forall G_n$ on $h \geq n_0$ vertices,

$$\text{gain}(M, G_n) \geq -\epsilon. \tag{27}$$

This means that beyond some n_0, we can do no worse than ϵ.

Definition 7.8 (Positive gain (PG))**.** Mechanism M satisfies the *positive gain* property if $\exists \gamma > 0, n_1 \in \mathbb{N}$: $\forall n \geq n_1, \exists G_n$ on n vertices such that

$$\text{gain}(M, G_n) \geq \gamma. \tag{28}$$

This means that there exist instances where we do better.

PG tells us that there are neighborhoods where delegation happens.

Theorem 7.9. *For any $\alpha_0 \in [0, 1)$, there is no local mechanism that satisfies the DNH and PG properties.*

Proof by illustration. We set up a graph where under direct democracy, the probability of getting a correct answer goes to 1.

In a liquid democracy, a constant fraction of the green vertices delegate to the blue vertices. There is some constant probability that *all* blue vertices vote *incorrectly* together. If this happens, then it is very likely that the majority is incorrect.

Thus there is a constant advantage between direct and liquid democracy. Thus this constant gap *does not close* and the graph grows larger and larger, which contradicts DNH. ☐

Note. The above construction is robust to model extensions where incompetent voters delegate to voters who are even more incompetent given their belief that the incorrect candidate is indeed correct.

7.2.3 Extensions

Delegating to less competent voters can be highly beneficial. If we consider a star with k leaves where the center has $p_i = 0.98$ and leaves have $p_i = 0.99$. We can add k isolated vertices with $p_i = 0$.

When all vertices vote independently, the probability of success $\to 0$ as $k \to \infty$, but when center votes for the entire star, the probability of success is 0.98.

7.3 The subjective model with optional participation

We consider an infinite population of voters given by a distribution μ over interval $[a, b]$, and a set N of n proxies with locations $\boldsymbol{x} \in [a, b]^n$ for large N.

Under a direct democracy, only the voters in N vote and we compute the median $\text{med}(\boldsymbol{x})$ or the mean $\text{mn}(\boldsymbol{x})$. In a liquid democracy, each voter in the population delegates to the closest proxy, leading to weights $\boldsymbol{w}$ and we compute the median $\text{med}(\boldsymbol{x}, \boldsymbol{w})$ or the mean $\text{mn}(\boldsymbol{x}, \boldsymbol{w})$.

Direct democracy is evaluated via

$$|\text{med}(\mu) - \text{med}(\boldsymbol{x})|, \qquad |\text{mn}(\mu) - \text{mn}(\boldsymbol{x})| \tag{29}$$

and liquid democracy is evaluated via

$$|\text{med}(\mu) - \text{med}(\boldsymbol{x}, \boldsymbol{w})|, \qquad |\text{mn}(\mu) - \text{mn}(\boldsymbol{x}, \boldsymbol{w})|. \tag{30}$$

Theorem 7.10. *For any $n \in \mathbb{N}, \boldsymbol{x} \in [a, b]^n$ and distribution μ,*

$$|\mu - \text{med}(\boldsymbol{x}, \boldsymbol{w})| \leq |\text{med}(\mu) - \text{med}(\boldsymbol{x})|. \tag{31}$$

Proof. Observe that $\text{med}(\boldsymbol{x}, \boldsymbol{w})$ is always the x_i that is closest to $\text{med}(\mu)$.

Let x_i be the closest proxy. Given $\text{med}(\mu)$, half of the weight is below and above this median. All of the weight in the interval $[x_i, \text{med}(\mu)]$ is assigned to x_3. □

Theorem 7.11. *Let $n = 2$. Then for any $\boldsymbol{x} \in [a, b]^n$ and distribution μ,*

$$|\text{mn}(\mu) - \text{mn}(\boldsymbol{x}, \boldsymbol{w})| \leq |\text{mn}(\mu) - \text{mn}(\boldsymbol{x})|. \tag{32}$$

The result doesn't hold for $n \geq 3$. We can construct an example where this is satisfied for $n = 1001$ in the lecture slides.

Note. If we suppose μ is the uniform distribution over $[a, b]$ and $x_1, \ldots, x_n$ are sampled independently from μ, we note that *both* $\text{mn}(\boldsymbol{x}), \text{mn}(\boldsymbol{x}, \boldsymbol{w}) \to \text{mn}(\mu)$ as $n \to \infty$.

We note that the weighted mean approaches faster than the unweighted mean

8 February 16th, 2022

Recall three different ballot types: rankings, approvals, and scores.

Today, we will discuss approval voting. This is a voting scheme where voters can approve as many alternatives as they like. We then elect the alternative that is approved by the most voters.

The motivation is that this incorporates many elements of plurality voting and gives voters expressive power. Other advantages include reducing the threat of splitting votes for similar candidates.

8.1 Approval-based committees

Denote the approved set of voter $i \in N$ by $\alpha_i \subseteq A$. The outcome is a committee $W \subseteq A : |W| = k$. We will assume utility of voter $i \in N$ for $W \subseteq A$ is

$$u_i(W) = |\alpha_i \cap W|, \tag{33}$$

that is, the utility is the number of approved alternatives that end up in the committee.

8.1.1 Thiele's methods

We want to discuss how to elect a committee given some approval votes.

Given a sequence $s_1, s_2, \dots$, we select a committee W that maximizes

$$\sum_{i \in N} (s_1 + s_2 + \cdots + s_{u_i(W)}). \tag{34}$$

We tend to want to maximize the sum of the utilities across all voters.

Example 8.1. Approval voting (AV), Chamberlin-Courant (CC), and proportional approval voting (PAV) are special cases of Thiele methods.

Note. Thiele himself proposed using the harmonic series as scores,

$$f(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \tag{35}$$

Example 8.2. We then give an example of forming a committee of 11 alternatives by maximizing the utilities from proportional approval voting. The result is that we are giving every party the number of seats that is proportional to its size.

8.1.2 Harmonic numbers

We seek to satisfy *proportionality*.

Definition 8.3 (Proportionality)**.** Supposing a party list has x supporters with $x \geq \ell \frac{n}{k}$ then it deserves ℓ seats. It holds that

$$\frac{x}{1} > \frac{x}{2} > \cdots > \frac{x}{\ell} \geq \frac{n}{k}. \tag{36}$$

We note that there cannot be more than k alternatives with marginal increase at least n/k. We note that k is the size of the committee and n is the number of voters.

8.2 Justified representation

We seek to define proportionality when approval sets *intersect*. We can consider *justified representation*.

Definition 8.4 (Justified representation (JR))**.** If there is $S \subseteq N : |S| \geq n/k$ and $|\cap_{i \in S} \alpha_i| \geq 1 \implies \exists i \in S : u_i(W) \geq 1$.

Note. Approval voting fails justified representation.

Theorem 8.5. *Chamberlin-Courant satisfies justified representation.*

Proof. Assume for contradiction that W is a CC committee violating justified representation and S be the subset witnessing the violation. The number of voters covered by W is less than $n \implies \exists x \in W$ whose marginal contribution is less than n/k voters.

We can remove x and add the candidate approved by S, which gives a greater CC score. □

We can extend this property:

Definition 8.6 (Extended justified representation (EJR))**.** If there is $S \subseteq N : |S| \geq \ell \frac{n}{k}$ and $|\cap_{i \in S} \alpha_i| \geq \ell \implies \exists i \in S : u_i(W) \geq \ell$.

Note. Chamberlin-Courant fails extended justified representation.

Note. Extended justified representation is stronger than justified representation, so approval voting also fails extended justified representation.

Note. In the party list setting, we note that extended justified representation $\implies$ proportionality.

Justified representation $\not\Rightarrow$ proportionality because having at least one voter is not enough for proportionality.

Theorem 8.7. *Proportional approval voting satisfies extended justified representation.*

Proof. Assume for contradiction that W is the PAV committee and suppose $S \subseteq N : |S| \geq \ell \frac{n}{k}$ and $|\cap_{i \in S} \alpha_i| \geq \ell$ but $u_i(W) < \ell \forall i \in S$.

Let $x^* \in \cap_{i \in S} A_i \setminus W, W' = W \cup \{x^*\}$. Then

$$\mathrm{PAV-score}(W') \geq \mathrm{PAV-W} + |S|\frac{1}{\ell} \geq \mathrm{PAV-score}(W) + \frac{n}{k}. \tag{37}$$

We claim that we can remove an alternative from W' and decrease PAV score by less than n/k. The average loss of PAV score after removal is

$$\frac{1}{k+1} \sum_{x \in W'} \sum_{i : x \in \alpha_i} \frac{1}{u_i(W')} = \frac{1}{k+1} \sum_{i \in N} \sum_{x \in \alpha_i \cap W'} \frac{1}{u_i(W')} = \frac{1}{k+1} \sum_{i \in N} 1 < \frac{n}{k}. \tag{38}$$

Therefore $\exists x' \in W$:

$$\mathrm{PAV-score}(W' \setminus \{x'\}) > \mathrm{PAV-score}(W), \tag{39}$$

which contradicts the optimality of W. □

Example 8.8. We then consider an example where PAV selects an outcome satisfying extended justified representation that doesn't *feel* proportional.

9 February 26th, 2022

Today, we will discuss participatory budgeting. We consider an allocation of a city's budget based on the votes of residents.

9.1 Participatory budgeting model

In this model, we have N voters, where each voter $i \in N$ cases an approval vote $\alpha_i \subseteq A$. Each $x \in A$ has cost $c(x)$ and there is a budget B. The outcome is a committee $W \subseteq A : c(W) = \sum_{x \in W} c(x) \leq B$.

We will assume for now that the utility for voter $i \in N$ for $W \subseteq A$ is $u_i(W) = |\alpha_i \cap W|$.

Note. A natural interpretation of approval voting is to maximize $\sum_{i \in N} u_i(W)$ subject to the budget constraint. The greedy algorithm is often used in practice by adding alternatives in order of approval score. This is similar to the *knapsack problem* we saw last semester.

Example 9.1 (Paris, 4th district 2019)**.** We can see an example of greedy allocation in Paris where the greedy algorithm can allocate to a very expensive project that takes up the budget. The optimal allocation funds all the small projects but not the large one.

Example 9.2 (Circleville)**.** We can consider an example where only one neighborhood's projects is funded because of the large population.

Recall that proportional allocation voting assigns a harmonic series of scores.

Example 9.3. We can consider an example of Rectangleville and New Rectangleville with a budget of 90. See the slides for more details.

Note. We note that PAV cannot distinguish between the two examples and cannot identify which outcomes are proportional.

Theorem 9.4. *Every voting rule that only depends on the collection of budget-feasible subsets must fail proportionality, even on instances with a district structure.*

9.2 Rule X

Consider the following rule.

We give a budget of B/n to each voter. Until the budget runs out, for each alternative, divide its cost as evenly as possible among its supporters. We then fund an affordable alternative with the lowest max payment.

We can define extended justified representation for participatory budgeting.

Definition 9.5 (Extended justified representation)**.** $\forall S \subseteq N : |S| \geq \gamma n$ and $T \subseteq \bigcap_{i \in S} \alpha_i : c(T) \leq \gamma B \implies \exists i \in S : u_i(W) \geq u_i(T)$.

Note. This is the same as normal EJR for committee elections if we take $B = k$ and $\gamma = \ell/k$.

Theorem 9.6. *Rule X satisfies EJR.*

9.3 A different perspective

We can consider general additive utilities

$$u_i(W) = \sum_{x \in W} u_i(x), \qquad u_i(x) \in \mathbb{R}^+. \tag{40}$$

The goal is to find $W \subseteq A$ that maximizes the social welfare

$$\text{sw}(W, \boldsymbol{u}) = \sum_{i \in N} u_i(W) \tag{41}$$

subject to $c(W) \leq B$.

Note. We don't want to ask voters to explicitly report utilities, but we will ask voters to cast votes in some *input format*.

9.3.1 Input formats

We can consider different projects with utilities and costs for a particular voter.

We can consider different input methods. We can *rank by value*. We can also *rank by value-for-money* (VFM). We can also do *knapsack voting* where a voter selects the projects that maximize their utility. The final idea is highly theoretical, called *threshold approval*, where voters approve projects that cost more than the threshold.

9.3.2 Implicit utilitarian voting

We can consider a model where voter i reports a vote σ_i that is *consistent with* u_i, denoted by $u_i \triangleright \sigma_i$. We consider a randomized voting rule f that maps a profile $\boldsymbol{\sigma}$ to a distribution over budget-feasible subsets of alternatives.

Definition 9.7 (Distortion)**.** The *distortion* of f on $\boldsymbol{\sigma}$ is

$$\max_{\boldsymbol{u} \triangleright \boldsymbol{\sigma}} \frac{\max_{W \subseteq A : c(W) \leq B} \text{sw}(W, \boldsymbol{u})}{E\left[\text{sw}(f(\boldsymbol{\sigma}), \boldsymbol{u})\right]}. \tag{42}$$

The distortion is capturing how badly the input format does at expressing the underlying utilities. We take the worst case over $\boldsymbol{u}$ of the optimal solution divided by the expected output.

We then associate an input format with the *worst-case* distortion of the *best* voting rule.

The theoretical distortion is:

Input format	Distortion
Value	$\tilde{\Theta}(\sqrt{m})$
VFM	$\tilde{\Theta}(\sqrt{m})$
Knapsack	$\Theta(m)$
Threshold	$O(\log^2 m)$

Table 4: Distortions for input formats.

We assume here that the overall utility we assign to voters is fixed.

10 February 28th, 2022

Today is a class-wide discussion of which approach (axiomatic, game-theoretic, epistemic) is relevant to the design of real-world democratic systems, and which voting rule (plurality, Borda, STV, Kemeny, Approval) to use in a mayoral election.

10.1 Relevance in practice

Which theoretical approaches are relevant in practice? We recall the different approaches.

The axiomatic approach involves May's theorem, JR, EJR, the core, etc. The game-theoretic approach includes our lectures on strategy-proofness, impossibility theorems, single-peaked preferences, etc. The epistemic approach includes the Condorcet and Mallows noise models, MLEs, liquid democracy.

The axiomatic approach is very rigid and guides us and helps makes principled decisions, but doesn't really explain outcomes. Our axioms only connect some profiles with results, but doesn't have much explanatory power.

11 March 2nd, 2022

Today, we discuss fair division. This time, our connections to democracy will be more direct, like congressional apportionment, redistricting, etc.

The first few lectures are devoted to fair division.

11.1 Cake cutting

The central problem is dividing a heterogenous divisible good between players with different preferences.

We can think of a cake as an interval $[0,1]$ and a set of players $N = \{1, n\}$. A piece of cake $X \subseteq [0,1]$ as a finite union of subintervals of $[0,1]$.

Each player $i \in N$ has a non-negative valuation V_i over pieces of cake. We will make these assumptions about the valuation functions:

1. **Additive.** For $X \cap Y = \varnothing$, we have
$$V_i(X) + V_i(Y) = V_i(X \cup Y) \tag{43}$$
2. **Normalized.** $\forall i \in N : V_i([0,1]) = 1$.
3. **Divisible.** $\forall \lambda \in [0,1]$ can cut $I' \subseteq I : V_i(I') = \lambda V_i(I)$.

Our goal is to find an allocation $A_1, \dots, A_n$ that satisfies:

- **Proportionality.** $\forall i \in N$,
$$V_i(A_i) \geq \frac{1}{n} \tag{44}$$
- **Envy-freeness (EF).** $\forall i, j \in N$
$$V_i(A_i) \geq V_i(A_j), \tag{45}$$
  that is, nobody is envious of other players.

Note. For $n = 2$, we note that these two rules are *equivalent*.

And for $n \geq 3$, we note that *envy-freeness* is stronger.

11.1.1 Cut-and-choose

Player 1 divides into two pieces $X, Y : V_1(X) = V_2(Y) = \frac{1}{2}$. Player 2 chooses preferred piece. This is EF and hence proportional.

11.2 Robertson-Webb model

We will now consider the complexity of cut-and-choose. This model tells us what algorithms are allowed to do and defines a new notion of complexity. Note that our conventional notion of complexity is not evaluative for such models.

The input size is n and there are two operations:

- $\text{Eval}_i(x, y)$ returns $V_i([x, y])$.
- $\text{Cut}_i(x, \alpha)$ returns $y : V_i([x, y]) = \alpha$.

Note. The cut and choose protocol requires **two operations**. Player 1 completes one cut query and player 2 completes one evaluation query.

11.3 Dubins-Spanier protocol

We will now discuss this game on n players. Recall that proportionality is a weaker property for $n \geq 3$.

In this protocol, a referee continuously moves a knife over a cake. We repeat the following: when a piece left of the knife is worth $1/n$ to a player, the player shouts "stop" and gets the piece. The player is removed and we repeat. The last player gets the remaining piece.

Note. This is proportional but not envy-free.

Note. The complexity is $\Theta(n^2)$.

11.4 Even-Paz protocol

Given $[x, y]$, assume $n = 2^k$ for ease of exposition. If $n = 1$m give $[x, y]$ to the single player. Otherwise, each player i makes a mark z_i :

$$V_i([x, z_i]) = \frac{1}{2} V_i([x, y]). \tag{46}$$

Let z^* be the $n/2$ mark from the left. We will recurse on $[x, z^*]$ with the left $n/2$ players and on $[z^*, y]$ with the right $n/2$ players.

Theorem 11.1. *The Even-Paz protocol produces a proportional allocation.*

Proof. At state 0, each of the n players value the whole cake at 1. At each stage, the players who share a piece of cake value it at least at $\frac{1}{2} V_i([x, y])$. Hence, at stage k each player has at least value $1/2^k$ for the piece they are sharing, then at stage $k + 1$, each player has value at least $1/2^{(k+1)}$. The number of stages is $\log n$. □

Claim (Even-Paz runtime)**.** The recurrence relation is $T(1) = 0, T(n) = 2n + 2T(n/2)$.

There are $\log n$ iterations and we pay $2n$ each time, so overall, the runtime is $\boxed{2n \log n.}$

Theorem 11.2. *Any proportional protocol needs $\Omega(n \log n)$ operations in the Robertson-Webb model.*

This means that the Even-Paz protocol is provably optimal.

We will now introduce an envy-free protocol for three players.

11.5 Selfridge-Conway procedure

There are three stages.

- **Stage 0.** Player 1 divides the cake into three equal pieces according to V_1. Player 2 trims the largest piece such that there is a tie between the two largest pieces according to V_2. Cake 1 is the cake without trimmings, cake 2 is the trimmings.
- **Stage 1.** Player 3 chooses one of the three pieces of cake 1. If player 3 did not choose the trimmed piece, player 2 is allocated the trimmed piece. Otherwise, player 2 chooses one of the two remaining pieces. Player 1 gets the remaining piece. We can denote the player $i \in \{2, 3\}$ that received the trimmed piece by T and the other by T'.
- **Stage 2.** T' divides cake 2 into three equal pieces according to $V_{T'}$. Players $T, 1, T'$ choose the pieces of cake 2, in that order.

Note. It is clear that Players 2 and 3 are not envious. Player 1 has an *irrevocable advantage over T.*

11.6 Complexity of EF

Theorem 11.3 (Brams and Taylor (1995))**.** *There is an EF cake cutting algorithm in the RW model, but it is unbounded.*

Theorem 11.4 (Aziz and Mackenzie (2016))**.** *There is a bounded EF algorithm for any n whose complexity is* $O(n^{n^{n^{n^{n^{n^{n}}}}}})$

Theorem 11.5 (Procaccia (2009))**.** *Any EF algorithm requires $\Omega(n^2)$ queries in the RW model.*

12 March 7th, 2022

13 March 9th, 2022

Announcements

- Today will be the last lecture with masks

13.1 Indivisible goods

We will start with model notation. We have a set G of m goods. Each good is *indivisible*. We have players $N = \{1, \ldots, n\}$ with valuations V_i over bundles of goods. We will assume that valuations are *additive* if $\forall S \subseteq G, i \in N : V_i(S) = \sum_{g \in S} V_i(g)$.

We define an *allocation* as a partition of goods $\boldsymbol{A} = (A_1, \ldots, A_n)$.

Note. Envy-freeness and proportionality are infeasible We will seek to define some fairness descriptions.

13.2 Maximin share guarantee

Definition 13.1 (Maximin share (MMS) guarantee)**.** The *maximin share guarantee* of player i is

$$\max_{X_1, \ldots, X_n} \min_j V_i(X_j) \tag{47}$$

An MMS allocation is such that $V_i(A_i)$ is at least as large as i's MMS guarantee for all $i \in N$.

Note. For $n = 2$ an MMS allocation always exists via *cut-and-choose.*

Theorem 13.2. $\forall n \geq 3 \exists$ *additive valuation functions that do not admit an MMS allocation.*

Example 13.3. For a counterexample for $n = 3$, we can construct a situation where there are three ways of dividing these numbers into three subsets of four numbers such that each subset adds to 55. These can be found in the slides.

We can construct valuation functions where each player has the same valuation function in their own partitions, but we cannot allocate these bundles in this way to satisfy all players.

13.3 Approximate envy-freeness

For this notion of fairness we assume general *monotonic* valuations. That is, $\forall S \subseteq T \subseteq G, V_i(S) \leq V_i(T)$.

Definition 13.4 (Envy-freeness up to one good (EF1))**.** An allocation $\boldsymbol{A}$ is *envy-free up to one good (EF1)* $\iff \forall i, j \in N, \exists g \in A_j : v_i(A_i) \geq v_i(A_j \setminus \{g\})$.

Theorem 13.5. *An EF1 allocation exists and can be found in polynomial time.*

We will prove the above theorem by considering *partial allocations* and the following lemma.

Definition 13.6 (Partial allocation)**.** A *partial allocation* is an allocation of a subset of the goods.

Lemma 13.7. *An EF1 partial allocation* $\boldsymbol{A}$ *can be transformed in polynomial time to an EF1 partial allocation* $\boldsymbol{B}$ *of the same goods with an acyclic envy graph.*

Proof. If the graph has a cycle C, we shift allocations along C to obtain $\boldsymbol{A'}$. This is clearly EF1. We also note that the number of edges in $\boldsymbol{A'}$ has strictly decreased. We have the same edges between $N \setminus C$. Edges from $N \setminus C$ to C are simply shifted. Edges from C to $N \setminus C$ can only decrease. Edges inside C decreased.

We can then iteratively remove cycles in polynomial time. ☐

We will now prove the theorem.

Proof. Given a partial allocation $\boldsymbol{A}$, we can construct an *envy-graph* where we create edge (i, j) to note that i envies j.

We can maintain EF1 and acyclic envy graph. In round 1, we allocate good g_1 to arbitrary player. The envy graph is still acyclic and EF1. We then allocate $g_1, \ldots, g_{k-1}$ in acyclic and Ef1 allocation $\boldsymbol{A}$. We can derive $\boldsymbol{B}$ by allocating g_k to *source* i.

We claim this allocation is still EF1 because no one envied i before and now they are allocated one good. Thus

$$V_j(B_j = V_j(A_j) \geq V_j(A_i) = V_j(B_i \setminus \{g_k\}). \tag{48}$$

Then $\boldsymbol{B}$ is EF1 and we can use the lemma to eliminate cycles, and we keep going until we eliminate all the cycles. □

13.4 Round robin

We will now return to *additive* valuations. Proving the existence of EF1 allocation is trivial.

Definition 13.8 (Round robin)**.** In a *round-robin* allocation, players take turns to pick items like an NBA draft.

Theorem 13.9. *A round-robin allocation is EF1.*

Proof. We can divide the round-robin allocation into phases, where each phase is defined by a player, say player i, picking an item. In each phase, i prefers their pick over all others. So i is envy-free across all items allocated across phases. There may be items chosen *before* the first phase that i envies, but these are all chosen exactly once by all players $j \neq i$. Thus we have EF1. □

Definition 13.10 (Pareto efficiency)**.** An allocation $\boldsymbol{A}$ is *Pareto efficient* if $\nexists \boldsymbol{A'} : V_i(A'_i) \geq V_i(A_i) \forall i \in N$ and $V_j(A'_j) > V_j(A_j)$ for some $j \in N$.

Note. Max utilitarian social welfare is Pareto efficient because if there was an $\boldsymbol{A'}$, this would increase social welfare and we can just choose it.

Round robin is **not** Pareto efficient.

13.5 Maximum Nash welfare

We want to define a notion of fairness that is *both* EF1 and Pareto efficient.

Definition 13.11 (Nash welfare)**.** The *Nash welfare* of an allocation $\boldsymbol{A}$ is

$$\mathrm{NW}(\boldsymbol{A}) = \prod_{i \in N} V_i(A_i). \tag{49}$$

The *maximum Nash welfare (MNW)* solution chooses an allocation that maximizes the Nash welfare.

We will ignore the case of $\mathrm{NW}(\boldsymbol{A}) = 0 \forall \boldsymbol{A}$.

Theorem 13.12. *Assuming additive valuations, the MNW solution is EF1 and Pareto efficient.*

Proof. Efficiency is trivial. We will focus on EF1. AFTSOC that i envies j by more than one good. Let $g^* \in \operatorname{argmin}_{g \in A_j} V_j(g)/V_i(g)$. We will move g^* from j to i to obtain $\boldsymbol{A'}$ and show that $\mathrm{NW}(\boldsymbol{A'}) > \mathrm{NW}(\boldsymbol{A})$.

Note that $V_k(A_k) = V_k(A'_k) \forall k \neq i, j$ and

$$V_i(A'_i) = V_i(A_i) + V_i(g^*), \qquad V_j(A'_j) = V_j(A_j) - V_j(g^*). \tag{50}$$

Then

$$\frac{\mathrm{NW}(\boldsymbol{A'})}{\mathrm{NW}(\boldsymbol{A})} > 1 \iff \left[1 - \frac{V_j(g^*)}{V_j(A_j)}\right]\left[1 + \frac{V_i(g^*)}{V_i(A_i)}\right] > 1 \iff \frac{V_i(g^*)}{V_j(g^*)}\left[V_i(A_i) + V_i(g^*)\right] < V_j(A_j). \tag{51}$$

Due to our choice of g^* and by violation of EF1, we have

$$\frac{V_j(g^*)}{V_i(g^*)} \leq \frac{\sum_{g \in A_j} V_j(g)}{\sum_{g \in A_j} V_i(g)} = \frac{V_j(A_j)}{V_i(A_j)}, \qquad V_i(A_i) + V_i(g^*) < V_i(A_j). \tag{52}$$

We can multiply the last two inequalities to get the first and we are done. □

13.6 An open problem

Definition 13.13 (Envy-free up to any good (EFX))**.** An allocation $\boldsymbol{A}$ is *envy-free up to any good (EFX)* $\iff \forall i, j \in N, g \in A_j$, we have $v_i(A_i) \geq v_i(A_j \setminus \{g\})$.

This is strictly stronger than EF1 and strictly weaker than EF.

An EFX allocation exists for two players with monotonic valuations (easy) and for three players with additive valuations (very hard).

Existence is an open problem for $n \geq 4$ players with additive valuations.

14 March 21st, 2022

Today, we discuss assignment problems. For example, consider school choice or housing allocation. In these cases, each player requires exactly one good.

14.1 Random assignment

We will consider players $N = \{1, \ldots, n\}$, a set G of n goods. Each player has ranking $\sigma_i \in \mathcal{L}$ over G. An *assignment* is a perfect matching π between players and goods where $\pi(i)$ is a good assigned to i.

We are interested in rules f that take $\boldsymbol{\sigma} \in \mathcal{L}^n$ and output π.

14.2 Serial dictatorship

In *serial dictatorship*, players select their favorite goods according to a predetermined order τ.

Definition 14.1 (Pareto efficient)**.** An assignment π is *Pareto efficient* if there is no assignment $\pi' : \pi'(i) \succeq_{\sigma_i} \pi(i) \forall i \in N$ and $\pi'(j) \succ_{\sigma_j} \pi(j)$ for some $j \in N$.

Definition 14.2 (strategy-proofness (SP))**.** A rule f is *strategy-proof* if for all $\boldsymbol{\sigma} \in \mathcal{L}^n$ for all $i \in N, \sigma_i' \in \mathcal{L}$:

$$f(\boldsymbol{\sigma})(i) \succeq_{\sigma_i} f(\sigma_i', \boldsymbol{\sigma}_{-i})(i). \tag{53}$$

Note. Serial dictatorship is both Pareto efficient and SP.

We also note that serial dictatorship is intuitively unfair for those who are late in the order. One way to circumvent this is to randomize the order. This produces *random serial dictatorship.*

14.3 Random serial dictatorship

Definition 14.3 (Random serial dictatorship (RSD))**.** *Random serial dictatorship* is serial dictatorship with the order τ chosen uniformly at random.

Definition 14.4 (Lottery)**.** A distribution over assignments is a *lottery.* A lottery induces a random assignment.

Note. A *random assignment* is a *bistochastic* matrix $P = [p_{ix}]$ where p_{ix} is the probably player i is assigned to x. A bistochastic matrix is one where the rows and columns sum to 1.

Definition 14.5 (Ex post strategy-proof)**.** A rule f is *ex post strategy-proof* if players cannot gain from lying regardless of random coin flips.

RSD is ex post strategy-proof. Moreover, RSD satisfies *equal treatment of equals.*

Definition 14.6 (Equal treatment of equals)**.** A rule f satisfies *equal treatment of equals* if $\forall i, j \in N : \sigma_i = \sigma_j, p_{ix} = p_{jx} \forall x \in G$.

RSD is also ex post Pareto efficient.

Definition 14.7 (Ex post Pareto efficient)**.** A rule f is *ex post Pareto efficient* if every assignment in the support is Pareto efficient.

14.4 Ordinal efficiency

We seek to define a notion of efficiency for lotteries.

Definition 14.8 (Stochastic domination)**.** A random assignment P' *stochastically dominates* $P \iff \forall i \in N, x \in G$:

$$\sum_{y \succeq_{\sigma_i} x} p'_{iy} \geq \sum_{y \succeq_{\sigma_i} x} p_{iy} \tag{54}$$

with at least one strict inequality.

Definition 14.9 (Ordinal efficient)**.** A random assignment is *ordinal efficient* if it is not stochastically dominated by any other assignment.

Note. We note that ordinal efficiency $\implies$ ex post efficiency.

14.5 Probabilistic serial rule (PS)

The *probabilistic serial rule* is directly defined by a random assignment. Each good is a *divisible* good consisting of probability shares. At every point in time, all players *eat* their favorite remaining goods at the same rate. When all goods are eaten, each player has probability shares adding up to 1.

Note. Probabailistic serial satisfies envy-freeness.

14.5.1 Random assignment to lottery

We will discuss if an arbitrary random assignment induces a lottery.

Definition 14.10 (Permutation matrix)**.** A *permutation matrix* is a bistochastic matrix consisting only of zeros and ones.

A permutation matrix represents an assignment due to the bistochastic property and elements are binary.

Theorem 14.11 (Birkhoff-von Neumann)**.** *Any bistochastic matrix can be obtained as a convex combination of permutation matrices.*

Note. PS satisfies equal treatment of equals.

Theorem 14.12. *PS is ordinally efficient.*

Proof. Given a random assignment P and a profile $\boldsymbol{\sigma}$, we define a graph $\Gamma_{P,\boldsymbol{\sigma}} = (G, E) : (x, y) \in E \iff \exists i \in N : x \succ_{\sigma_i} y$ and $p_{iy} > 0$.

Lemma 14.13. *If* $\Gamma_{P,\boldsymbol{\sigma}}$ *is acyclic then* P *is ordinally efficient.*

If P is output of PS, claim that $\Gamma_{P,\boldsymbol{\sigma}}$ is acyclic and conclude by the lemma. Assume for contradiction $\Gamma_{P,\boldsymbol{\sigma}}$ has a cycle. Let x be the first good in the cylce to be fully eaten at time t. Then $\exists (y, x) \in \Gamma_{P,\boldsymbol{\sigma}} \implies \exists i \in N : y \succ_i x$ and $p_{ix} > 0$.

But note that at any point up to t, i should have been eating y or a more preferred good, which contradicts $p_{ix} > 0$. □

Note. PS is **not** strategy-proof by example.

14.6 Impossibility

Theorem 14.14. *There is no rule that satisfies ordinal efficiency, strategy-proofness, and equal treatment of equals.*

14.6.1 Empirical and theoretical results

In 2006, Pathak ran RSD and PS on truthful data from 8255 students in New York City. He found that the two mechanisms were essentially equivalent in assigning the same distribution across students and preferences. At every prefix, PS has a *slight advantage* over RSD.

In 2010, Che and Kojima formalized this equivalence in large instances between RSD and PS. The two random assignments converge to the same limit.

Because of these results, RSD is preferred over PS due to strategy-proofness. The theoretic advantage of PS over RSD is extremely small and is not robust to manipulation/deviation.

15 March 23rd, 2022

Today we will discuss *sortition* — which is democracy built on lotteries instead of elections.

15.1 Sortition

Sortition was first used in Athens in the Council of 500 and magisracies chosen by lot. It was used in Florence in 1328-1530 when government and legislative councils were chosen by lot. And since 1776, the United States made democracy synonymous with elections — a pretty recent development. In modern times, sortition is not used except for citizens' assemblies organized by local and national governments.

Example 15.1 (2020 Climate Assembly UK)**.** Sortition was used to create a citizens' assembly to discuss climate change in the United Kingdom.

There are many aspects of sortition to discuss, but today, we will focus on the process by which we select our assemblies.

In the ideal pipeline, if we want an assembly size k, we just pick k people uniformly at random from the population. The actual pipeline is as follows. From the population, we send letters to *letter recipients*. People who receive the letter and agree to participate will create a *pool*. The panel is then chosen from this pool uniformly at random.

Note. We want the panel to be representative of the general population. We note that the set of letter recipients is representative but the set that creates the pool is highly unrepresentative because of self-selection bias.

Example 15.2 (Climate Assembly UK)**.** We note that for this assembly, the education and climate features of the pool is *not representative* of the general populace.

We can choose a representative panel at random from a pool of highly unrepresentative citizens using *quotas*.

15.2 Sortition model

We consider a set of *features* F where each $f \in F$ has a set of *values* V_f. A multiset of n volunteers N where each $\boldsymbol{x} \in N$ is a vector of feature values. For each $f \in F, v \in V_f$ there is an *upper quota* $u_{f,v}$ and a *lower quota* $\ell_{f,v}$. The goal is to choose a *panel* P of k volunteers such that for all $f \in F, v \in V_f, \ell_{f,v} \leq |\{\boldsymbol{x} \in P : x_f = v\}| \leq u_{f,v}$.

Note. Finding a quota-feasible panel is NP-hard.

15.2.1 A greedy algorithm

At a time t there is a partial panel P_t that is selected. Note $P_0 = \varnothing$. For each $f \in F, v \in V_f$, define the score of v to be

$$\frac{\ell_{f,v} - |\{\boldsymbol{x} \in P_t : x_f = v\}|}{|\{\boldsymbol{x} \in N \setminus P_t : x_f = v\}|}. \tag{55}$$

Intuitively, the numerator is the number of people we still need to recruit with the feature value in order to satisfy the lower quota. The denominator is the number of volunteers remaining in the pool that satisfy the feature value.

For v with maximum score, we select uniformly at random among $\boldsymbol{x} \in N \setminus P_t : x_f = v$. When all lower quotas have been filled, select uniformly at random among $N \setminus P_t$. If any quotas cannot be satisfied, restart.

Example 15.3 (Greedy algorithm)**.** We consider an example. See slides.

Note. Looking at the distribution of probabilities of being chosen, most of the volunteers have 0 probability of being chosen.

We can consider a distribution over panels in the example. We see that with different quotas, different arrangements of people in our panel are more likely than others.

This motivates us to consider a general approach to try to come up with fair outcomes for sortition. We will draw inspiration from fair division. Thus we will think of a distribution over panels size k as *dividing the overall selection probability of k between pool members.*

15.2.2 Fair division and allocation rules

An allocation rule outputs a distribution $\mathcal{D}$ over quota-feasible panels size k given our input.

Definition 15.4 (Maximum Nash welfare (MNW))**.** Recall that *maximum Nash welfare* maximizes the product

$$\prod_{\boldsymbol{x} \in N} p_{P\sim\mathcal{D}}(\boldsymbol{x} \in P) \tag{56}$$

Definition 15.5 (Leximin)**.** *Leximin* selects an outcome that maximizes $\min_{\boldsymbol{x}\in N} p_{P\sim\mathcal{D}}(\boldsymbol{x} \in P)$ among all outcomes that maximizes the second lowest utility, among all outcomes that maximizes the third lowest utility, etc.

In our context, it will choose the distribution over feasible panels that maximizes the lowest probability of any volunteer out of all such distributions that maximizes the second lowest probability and so on and so forth.

Note. We note that both MNW and Leximin equalizes volunteers' selection probabilities whenever the quotas make it feasible to do so.

For MNW, we can intuitively think of a simple problem of having $x, 1-x$ and take the product. This product is maximized when $x = \frac{1}{2}$.

For Leximin, if we can equalize probabilities, this means that each player has probability k/n. If we did not equalize probability, someone will have probability $< k/n$ and this is suboptimal in Leximin because the minimum will be lower. So the only way to get the minimum to k/n is if everyone has probability k/n.

Note. Computing distributions is very very difficult.

Example 15.6 (Panelot)**.** Panelot takes inputs as we've described and returns a Leximin distribution over feasible panels.

We need a uniform distribution over a given number of panels. If we want a panel of k, each panel should have a probability of $c/k, c \in \mathbb{N}$. This is a constraint on the optimization.

15.3 End-to-end guarantees

Denote the population size by m, the number of letter recipients by r, and the panel size k.

The probability someone from the population makes their way to letter recipients is r/m. For every individual, there is some probability of accepting an invitation if they get one $q(\boldsymbol{x})$. This is the opt-in probability. We can then select people from the pool to the panel in a way that is proportional to their opt-in probabilities ($\propto 1/q(\boldsymbol{x})$). If we have this, then the overall probability of people making it from the population to the panel is $\approx k/m$.

We can thus implement this ideal sortition pipeline.

15.3.1 Formalism

Let M be the population $|M| = m$ and r the number of letters sent. Let $m_{f,v} = |\{\boldsymbol{x} \in M : x_f = v\}|$ and let $q : \prod_{f \in F} V_f \to [0,1]$ give the opt-in probability of each $\boldsymbol{x} \in M$ and $\alpha = \min_{\boldsymbol{x} \in M} q(\boldsymbol{x}) \times r/k$.

Theorem 15.7. *Suppose that $\alpha \to \infty, m_{f,v} \geq m/k$ for all $f \in F, v \in V_f$. Then $\exists$ an allocation rule such that $\forall \boldsymbol{x} \in M$*

$$p(\boldsymbol{x} \in P) \geq (1 - o(1))\frac{k}{m} \tag{57}$$

and the quotas

$$\ell_{f,v} = (1 - o(1))\frac{km_{f,v}}{m} - |F|, \qquad u_{f,v} = (1 + o(1))\frac{km_{f,v}}{m} + |F| \tag{58}$$

are satisfied for all $f \in F, v \in V_f$.

Thus we get our end-to-end guarantee and we get representation for future values ex-post with high probability. Intuitively, everyone has the *right* selection probabilities.

16 March 28th, 2022

17 March 30th, 2022

Today, we discuss apportionment in the 20th century.

17.1 Apportionment in the 20th century

Recall our model: We have a set of states $N = \{1, \ldots, n\}$ with K seats to be allocated. Each state has population p_i and total population $P = \sum_i p_i$. The *standard quota* of state i -s $q_i = p_i/P \cdot K$. The *upper quota* of i is $\lceil q_i \rceil$ and ti *lower quota* is $\lfloor q_i \rfloor$. Let k_i be the number of seats allocated to i.

17.2 Huntington-Hill method

We define the founding function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x < \sqrt{\lfloor x \rfloor \cdot \lceil x \rceil} \\ \lceil x \rceil, & x \geq \sqrt{\lfloor x \rfloor \cdot \lceil x \rceil} \end{cases} \tag{59}$$

This method takes the desired number of seats K. It finds a divisor $D : \sum_i f(\hat{q}_i) = K$ where $\hat{q}_i = p_i/D$ is the modified quota. Each state is allocated $k_i = f(\hat{q}_i)$.

17.2.1 Divisor methods

By changing our rounding function f, we can obtain a family of apportionment methods called *divisor methods.* f must satisfy: $x \in \mathbb{Z} \implies f(x) = x$ and $x \geq y \implies f(x) \geq f(y)$.

Note for rounding functions, we have the following theorem:

Theorem 17.1. *Fixing f, if $D \neq D'$ that yield apportionments $k_1, \ldots, k_n$ and $k'_1, \ldots, k'_n$ then $k_i = k'_i \forall i \in N$.*

From Jefferson. Assume WLOG $D \leq D'$. Then $p_i/D \geq p_i/D' \forall i \in N$. We conclude $k_i = f(p_i/D) \geq f(p_i/D') = k_i; \forall i \in N$. It also holds that $\sum_{i \in N} k_i = K = \sum_{i \in N} k'_i$. Then It cannot be the case that $k_i > k'_i$ for some $i \in N$. ☐

Theorem 17.2. *A divisor method is the Huntington-Hill method $\iff \forall i, j \in N : p_i/k_i \leq p_j/k_j$,*

$$\frac{p_i/k_i}{p_j/k_j} > \frac{p_j/(k_j+1)}{p_i/(k_i-1)}. \tag{60}$$

Intuitively, when we transfer a seat from j to i, it becomes more inequitable.

Only if. The modified quota $\hat{q}_i = p_i/D$ is rounded down to k_i when $k_i \leq p_i/D < \sqrt{l_i(k_i+1)}$ and rounded up to k_i when $k_i \geq p_i/D \geq \sqrt{k_i(k_i-1)}$. Then

$$\sqrt{k_i(k_i-1)} \leq p_i/D < \sqrt{k_i(k_i+1)} \iff \frac{k_i(k_i-1)}{p_i^2} \leq \frac{1}{D^2} < \frac{k_i(k_i+1)}{p_i^2} \iff \frac{k_i(k_i-1)}{p_i^2} < \frac{k_i(k_i+1)}{p_i^2} \tag{61}$$

which is equivalent to the desired property. ☐

Note. Joseph A. Hill initially suggested a method based on minimizing the *relative differences* in citizens per seat. Edward V. Huntington, a Harvard math professor, formalized the idea and showed its equivalence to rounding at the geometric mean. This shows that the method slightly favors small states.

Some more history:

- In 1921 Congress considered bills based on Webster and Huntington-Hill, but both were rejected.
- In 1929 Congress turned to the National Academy of Sciences. The committee favored Huntington-Hill because it minimizes relative differences and because it *occupies mathematically a neutral position with respect to the emphasis on larger and smaller states.*
- In 1930, there was no disagreement between Webster and Huntington-Hill and apportionment was enacted.
- In 1940, Huntington-Hill gave Arkansas an extra seat and Webster gave Michigan an extra seat. Arkansas was Democratic and Michigan was Republican, so this became partisan.
- In 1941, President Roosevelt signed into law designating Huntington-Hill as the permanent apportionment method.

17.3 Properties revisited

There are two properties that we discussed previously and are *violated* by Hamilton's method.

17.3.1 Population monotonicity

Definition 17.3 (Population monotonicity)**.** Suppose there are two censusues where populations in the second are denoted $p'_1, \dots, p'_n$ and apportionment by $k'_1, \dots, k'_n$. An apportionment method is *population monotonic* if $k_i < k'_i$ *and* $k_j > k'_j \implies p_i < p'_i$ or $p_j > p'_j$.

Theorem 17.4. *All divisor methods are population monotonic.*

Proof. Suppose $k_i < k'_i, k_j > k'_j$. Then $p_i/D < p'_i/D/$ and $p_j/D > p'_j/D'$. Then

$$p'_i > \frac{D'}{D}p_i, \qquad p'_j > \frac{D'}{D}p_j. \tag{62}$$

Note $D'/D \leq 1 \implies p'_j < p_j$ and $D'/D \geq 1 \implies p'_i > p_i$. □

17.3.2 House monotonicity

Definition 17.5 (House monotonicity)**.** An apportionment method is *house monotonic* if $K' > K$ with all other variables unchanged $\implies k'_i \geq k_i \forall i \in N$.

Theorem 17.6. *Any population monotonic apportionment method is house monotonic.*

Proof. Let $K' > K, p_i = p'_i \forall i \in N$. Let $j : k'_j > k_j$. For all $i \neq j$, if $k'_i < k_i$ then population monotonicity $\implies p_{j'} > p_j$ or $p'_i < p_i$ which is false. Then $k'_i \geq k_i \forall i \in N$. □

Corollary 17.6.1. *All divisor methods are house monotonic.*

17.3.3 The quota criterion

Definition 17.7 (Quota criterion)**.** An apportionment method satisfies the *quota criterion* if $\forall i \in N$,

$$\lfloor q_i \rfloor \leq k_i \leq \lceil q_i \rceil \tag{63}$$

Note. Of the five methods discussed (Hamilton, Jefferson, Adams, Webster, Huntington-Hill), **only Hamilton** satisfies the quota criterion.

17.4 Impossibility

Definition 17.8 (Neutrality)**.** An apportionment method is *neutral* if permuting the states permutes the seat allocation.

Theorem 17.9. *There is no apportionment method that is neutral, population monotonic, and satisfies the quota criterion.*

We will first prove a lemma:

Lemma 17.10. *Any method that satisfies all three properties satisfies the order-preserving property:* $p_j > p_i \implies k_j \geq k_i$.

Proof. We define an instance with $p'_i = p_j, p'_j = p_i$ and $p_t = p'_t \forall t \neq i, j$. By population monotonicity, wither $k'_i \geq k_i$ or $k'_j \leq k_j$. By neutrality, $k'_i = k_j, k'_j = k_i \implies k_j \geq k_i$. □

Proof. Assume we have an apportionment method that satisfies all three properties. We can construct an example where $k'_1 > k_1, k'_2 < k_2$ yet $p'_1 < p_1$ and $p'_2 > p_2$. □

Corollary 17.10.1. *No divisor method satisfies the quota criterion.*

17.5 Randomized apportionment

We can now consider randomized apportionment methods. Consider the following algorithm:

1. Take a random permutation of the label of the states
2. Provisionally allocate $\lfloor q_i \rfloor$ seats to each state $i \in N$ and let $r_i = q_i - \lfloor q_i \rfloor$
3. Draw $U \sim \mathcal{U}(0, 1)$
4. Let $Q_i = U + \sum_{j=1}^{i} r_i$
5. For each $i \in N$, we allocate an extra seat to state i if $[Q_{i-1}, Q_i)$ contains an integer

Note. The randomized apportionment algorithm gives each state its standard quota in expectation. It follows that its population monotonic in expectation. If satisfies the quota criterion ex post.

18 April 4th, 2022

Today, the guest lecturer is Jamie Tucker-Foltz. We will discuss redistricting as cake-cutting.

18.1 Redistricting

Redistricting is not really specified in the Constitution.

Many parties engage in *gerrymandering.*

Definition 18.1 (Gerrymander)**.** *Gerrymandering* is the division or arrangement of a territorial unit into election districts in a way that gives one political party an unfair advantage.

Note. This is a big problem for democracy.

There are many ideas to prevent it:

- Have an independent commission draw fair districts
- Use an interactive protocol (IP) with participation from both parties
- Statistically prove a map is gerrymandered

Today, we will focus on the second point.

18.1.1 Abstract model

We are given some set S, a state, and a set of feasible districts $\mathcal{D} \subseteq 2^S$. We have a set of parties $N = \{1, 2, \ldots, n\}$. We have some population measure $\mu : \mathcal{D} \to \mathbb{R}_{\geq 0}$ which gives us the population of people in each district. For each $j \in N$ the distribution function $v^j : \mathcal{D} \to \mathbb{R}_{\geq 0}$ which gives us the distribution. We are finally given a target number of districts $m \in \mathbb{Z}_{\geq 0}$.

Definition 18.2 (Partition)**.** A *partition* is a set P of m disjoint districts covering S each of equal measure.

The utility of party j is

$$u^j(P) = \left|\left\{D \in P : \forall i \neq j, v^j(D) > v^i(D)\right\}\right|. \tag{64}$$

18.1.2 Discrete graph model

We can also consider a graph G of indivisible census blocks. The we have

- $S = V(G)$ the state is the vertices of the graph
- $\mathcal{D} = \{D \subseteq S : \text{induced subgraph of } D \text{ is connected}\}$
- $\mu(D) = \sum_{b \in D}$ population of b
- $v^j(D) = \sum_{b \in D}$ number of j-voters in b

18.1.3 Geometry-free model

This is a continuous model of placing voters in buckets with no constraints. We have

- $S = \bigcup_{j \in N} V_j, V_j = (j, [0, p_j]), \sum_{j \in N} p_j = 1$
- $\mathcal{D} = \left\{\bigcup_{j \in N} (j, [a_j, b_j]) : \forall j \in N, 0 \leq a_j \leq b_j \leq p_j\right\}$
- $\mu(D) = \sum_{j \in N} (b_j - a_j)$
- $v^j(D) = b_j - a_j$

18.2 Fairness

We now discuss some notions of fairness.

18.2.1 Proportionality

One basic notion is proportionality. For all $j \in N$:

$$u^j(P) \geq \left\lfloor m \frac{v^j(S)}{\sum_{i \in N} v^i(S)} \right\rfloor. \tag{65}$$

18.2.2 Geometric target

Another way to think about proportionality is the *geometric target.* For all $j \in N$ let $P^j_{\max}$ be a partition maximizing u^j and let $P^j_{\min}$ be a partition minimizing u^j. Then

$$u^j(P) \geq \left\lfloor \frac{u^j(P^j_{\max}) + u^j(P^j_{\min})}{2} \right\rfloor. \tag{66}$$

Theorem 18.3. *In the geometry-free model, a partition satisfies proportionality $\iff$ it satisfies the geometric target (up to ties).*

Note. The intuition is that for the minority party, $u^j(P^j_{\min}) = 0$ and $u^j(P^j_{\max})$ is on average double the actual utility.

This motivates us to use the geometric target as a good fairness notion because we are doing the best we can given geometric constraints.

18.3 Fair division protocols

18.3.1 Landau-Reid-Yershov (LRY) protocol

This is an interactive protocol by Landau, Reid, and Yershov that uses a neutral administrator.

The protocol is given by the following:

1. An administrator presents both parties with a series of bipartitions $(L_1, R_1), (L_2, R_2), \ldots, (L_{m-1}, R_{m-1})$ of $S : L_i \subseteq L_{i+1}$.
2. For each $i \in m$, each party is asked, "Would you rather redistrict L_i with the other party redistricting R_i or vice versa?
3. Try to find an i such that one party prefers to redistrict L_i and the other prefers R_i. It no i exists, randomly select an outcome at the cross-over point.

This relies on the following theorem.

Theorem 18.4 (Good choice property)**.** *Restricting the feasible set of partitions to respect a given split, a party's preferred choice satisfies its geometric target.*

Note. LRY is realistically implementable, it has simply party participation, and is guaranteed to be within 2 districts of proportional/geometric target in the geometry-free model.

However, this relies heavily on the neutrality of the administrator, and this can be arbitrarily far from geometric target in grid-based model.

18.3.2 Cut and freeze

This protocol is by Pegden and Procaccia: partition, freeze, and re-partition until all districts are frozen.

Theorem 18.5. *In the geometry-free model, under optimal play, each party can guarantee a number of seats as in the following graphs:*

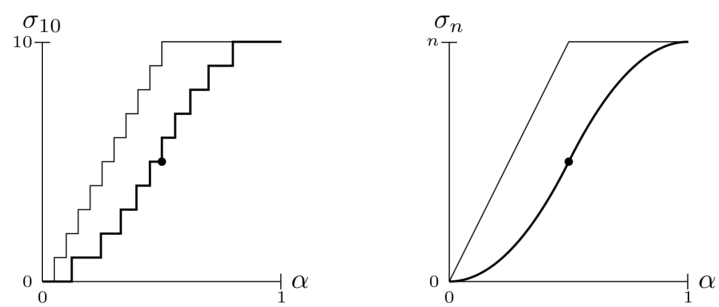

Figure 2: Cut and freeze for 10 districts.

Note. This protocol is realistically implementable, it approximates proportionality in geometry-free model, and it is hard to pack specific groups into one district.

However, this requires complicated strategies and requires several rounds of interaction.

18.3.3 State-cutting model 1

This is a cake-cutting analogue introduced by Benade, Procaccia, and Tucker-Foltz.

We have the following:

- $S = [0, 1]$
- $\mathcal{D} = \{\text{finite unions of closed intervals}\}$
- $\mu =$ Lebesgue measure
- $v^j(D) = \int_D f^j(D)$ where $\forall x \in S : \sum_{j \in N} f^j(x) = 1$.

The protocol is as follows:

1. Ask each party j to construct an optimal partition P_j
2. Construct a sequence of partitions from $P_1 \to P_2$ each differing from the previous one on at most two districts
3. Select an intermediate partition that satisfies the geometric targets of both parties.

Note. We can achieve step 2 using bubble sort. We can transition from $P_1 \to P_2$ via the simplest possible partition $\left\{ \left[\frac{k-1}{m}, \frac{k}{m} \right] : k \in [m] \right\}$.

Theorem 18.6. *If two partitions differ on at most two districts, the balance of power can differ by at most one.*

Proof. Suppose P, P' differ on districts $D_1, D_2 \in P, D'_1, D'_2 \in P'$. Suppose party 1 has a majority in D_1, D_2 but a minority in D'_1, D'_2. Then

$$\frac{1}{m} < v^1(D_1 + v^1(D_2) = v^1(D_1 \cup D_2) = v^1(D'_1 \cup D'_2) = v^1(D'_1) + v^1(D'_2) < \frac{1}{m}, \tag{67}$$

which is a contradiction. □

18.3.4 State-cutting model 2

In this modified model, parties are allowed to disagree over the distribution of voters!

The parameters are the same but we make one modification to the distribution function:

$$v_i^j(D) = \int_D f_i^j(D) \tag{68}$$

where $\forall x \in S, i \in N, \sum_{j \in N} f_i^j(x) = 1$.

Theorem 18.7. *Even when parties disagree, there always exists a partition satisfying the geometric targets of both parties:*

$$u_i^i(P) \geq \left\lfloor \frac{\min_{P'} u_i^i(P') + \max_{P'} u_i^i(P')}{2} \right\rfloor . \tag{69}$$

The protocol is as follows:

1. Each party i computes a maximal set $X_i \subseteq S : \mu(X_i) \in \mathbb{Z}, v_i^i(X_i) = \frac{1}{2}\mu(X_i)$.
2. Let i be the party with the larger X_i set and let j be the other party.
3. Party j divides X_j into two pieces of equal size and equal party support *according to* j.
4. Party i chooses a piece for j to redistrict.
5. Party i redistricts the rest of S.

Note. The state-cutting protocols guarantee geometric target in state-cutting model and works when parties disagree substantially over how voters are distributed.

However, protocols are both somewhat specific to the state-cutting model.

18.3.5 Threshold election

We now consider a simple asymmetric protocol with a non-standard election.

The protocol is

1. Party 1 constructs the partition
2. Party 2 observes it a chooses $m \in [0.5, 1)$
3. A *threshold election* is held in each district. If either party gets a majority strictly greater than m, they win. Otherwise, award the district randomly with equal probabilities for each party.

Note. This protocol has simple rules and strategies, guarantees expected proportionality in the geometry-free model, and the randomness is not required at equilibrium.

However, this is not realistically implementable, changes the entire election system, and we need to modify this substantially to get proportionality in other models.

19 April 6th, 2022

Today's guest lecturer is Professor Moon Duchin of the MGGG Redistricting Lab at Tufts University.

19.1 Gerrymandering

Every ten years, there is a Census where we apportion the 435 house seats to the states by population.

We have a few criterion: compactness, communities are connected, contiguity, the Voting Rights Act.

Example 19.1 (Illinois 4th district)**.** We can consider the *earmuff district* in Illinois that encapsulates Pilsen and Humboldt Park.

This was carved by Civil Rights activists to consolidate the Latino vote and unite them with three Black districts.

Definition 19.2 (Packing)**.** *Packing* a district refers to grouping all ethnic minorities together to create fewer minority districts.

There are many stakeholders in the gerrymandering problem. From a computational perspective, we can think of minimizing the loss, but this is too simple given the real-world. Today, we consider the detection problem of detecting gerrymandering.

19.2 Naive flags

The first naive flag is a bad shape. The second is disproportion. The former is more intuitive and the latter is about having equal representation in government.

Bad shapes only flag a *story*, like the earmuff district. A requirement for good shapes is not very constraining — we can create some simple pretty shapes that are just as bad for representation. Moreover, disproportion does **not** come for free; proportionality can be hard or impossible to achieve.

We want to create a test that flags abuse but leaves room for human discretion.

19.3 Formalism

We can view districts as *graph* partitions. We can connect atoms of our districts (counties, Census tracts, etc.) and think about this as a tiling of the state. We can then form a dual graph and partition into four connected subgraphs of equal population.

This problem is fairly hairy. For a 4×4 problem, there are 117 configurations with some dihedral symmetry — this is combinatorically hairy.

Example 19.3 (Pennsylvania)**.** Pennsylvania has

19.4 Sampling method

Example 19.4. Consider yellow and pink land where there are 40% orange in the entire 10×10 grid. If we draw from the distribution of partitions uniformly at random, we want to know how many districts we expect yellow to win.

We consider a sampling method. We explore with Markov chain that draws samples from the universe of compliant plans. Recall Markov chains are random walks with no memory.

Markov chains have a property that under conditions of ergodicity, Markov chain steps approach some stationary distribution where the probability distribution will stay the same when we take additional steps.

Definition 19.5 (Swap)**.** We can consider each step as a *swap* — where we swap two adjacent atoms in each step.

Example 19.6. If we sample using the *swap distribution* where we keep track of how many yellow districts there are, we see that proportionality where yellow gets four seats is an *outlier*. 40% of the voters will secure about 23% of the seats.

This 23% is fairly robust to what distribution we are sampling from.

19.5 Distributional design

We can create ensembles with prescribed characteristics in several ways: constraints; preferential acceptance.

This is a model where we accept with probability 1 new partitions with *lower* cut edges and accept with probability p new partitions with *greater* cut edges.

Example 19.7 (Metropolis in action)**.** Professor Duchin's undergraduate thesis advisor received an encoded letter from a prison. He used the Metropolis–Hastings algorithm to map the code space to the usual alphabet, using a corpus of all words in War and Peace.

This gets us to a decoding surprisingly fast. The decoding of the prison letter worked!

19.6 Current research

The central question is how long until the random walk gets a *representative sample*?

In a path with N nodes, the mixing time is N^2. For a grid with N nodes, the mixing time is N. In a hypercube of binary strings of N types, the mixing time is $\log N \log\log N$.

For grids, we don't know that much about swap chains and connectivity.

19.6.1 Swap chains

In Professor Duchin's research, she could not find swap and flip chains that behaved well.

We can think about *heating* and *cooling* to induce flips and steps (similar to the Ising model in physics) and note that when we do this to our metagraphs, we converge on districts that are similar to where we started.

19.6.2 Recombination

In the *recombination* algorithm, we take two districts, fuse them, draw a random spanning tree, and seek a balanced cut that partitions the graph into two subgraphs of equal population.

This is *incredibly fast* and the code is open-source.

Note (Flip algorithm)**.** Flip has some pros: tied to statistical physics literature, easy to explain, easy to target with Metropolis.

Flip has some cons: insanely slow-mixing and supported fractals. It also requires tons of user choices.

Note (ReCom)**.** ReCom approximately targets the spanning tree distribution.

There is a *reversible ReCom* that *directly* targets the spanning distribution and is within a slow-down factor of 1000 of the usual ReCom.

19.6.3 Open questions

The isospanimetric conjecture. This asks if on the open lattice, if we get a number of large vertices, we want to know which graph has the most spanning trees.

Balanced cuts of trees. What are the statistics of trees that *can* be cut in half.

Ergodicity, diameter bounds for metagraph, curvature of state space. On the triangle lattice, we need a slack of ideal size ± 1 for the three-district case.

Rapid mixing for ReCom on $n \times n$ grids, k districts.

19.6.4 New approaches

Other teams have been working on new approaches.

A team at Cornell is working on *fairmandering*, which is exponential branching that produces a *lot* of partitions. They do not know how their partitions are distributed on the state space.

A team at Harvard is doing sequential Monte Carlo where they recursively partition with weights.

19.7 Ripped from the headlines

Moon Duchin has been asked to consult on redistricting maps in Pennsylvania and North Carolina.

In North Carolina, Duchin created a set of decent maps but not a good set of principles.

20 April 11th, 2022

Today, we discuss weighted voting and the Electoral College. We have previously discussed voting mechanisms where all voters have equal weight. In some cases, it is desirable to have weighted voting.

20.1 Weighted voting games

In the Electoral College, each of the 50 states and DC is seen as a weighted voter with weight equal to the state's members of Congress. This assumes that electors of each state vote as a bloc and ignores special rules in Maine and Nebraska.

Definition 20.1 (Simple cooperative game)**.** A *simple cooperative game* is a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \{0, 1\}$ is the value function that assigns a value to all coalitions of players.

We assume that if $v(S) = 1, S \subseteq T \implies v(T) = 1$. Values are monotonic over coalitions.

A coalition with value 1 is a *winning coalition* and a coalition with value 0 is a *losing coalition*.

Definition 20.2 (Weighted voting game)**.** A *weighted voting game* is the simple cooperative game defined by a quota q and weight $w_i \in \mathbb{N} \cup \{0\}$ for all $i \in N$ where $\forall S \subseteq N, v(S) = 1 \iff \sum_{i \in S} w_i \geq q$.

Example 20.3. In a simple cooperative game with $n = 4$ where winning coalitions are $\{1, 2\}, \{3, 4\}, \{1, 2, 3\}$, $\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$. This is a valid simple cooperative game.

This cannot be represented as a weighted voting game because of the coalitions size 2. In each of the coalitions, one player must carry at least half the weight of the quota. Thus a quota with these two players should also be winning.

Definition 20.4 (Dummy player)**.** A *dummy player* is $i \in N : \forall S \subseteq N \setminus \{i\}, v(S) = v(S \cup \{i\})$.

Note. Note that the following conditions are sufficient, but not *necessary*, conditions for the existence of a dummy player in a weighted voting game:

- $\exists i \in N : w_i = 0$
- $\exists i \in N : w_i \geq q$ and $\sum_{j \neq i} w_j < q$

These are not necessary conditions given by the following example: $q = 4$, voter values $v_1 = 2, v_2 = 2, v_3 = 1$. Note that player 3 is a dummy player.

Example 20.5 (The common market)**.** The common market was formed in 1958 as a federation of six European countries. It is governed by the council of ministers which induces a weighted voting game with $q = 12$ and weights. France, Germany, Italy: 4; Belgium, Netherlands : 2; Luxembourg : 1.

Luxembourg is a dummy player!

Example 20.6 (UN security council)**.** A measure in the UNSC passes if 9 out of 15 members of the UNSC vote in favor, provided that no permanent member votes against it. This is a weighted voting game with $q = 49$, $w_i = 9$ for each permanent member and $w_i = 1$ for each non-permanent member.

20.2 Voting power

In weighted voting, the justification for weights is that voters have different degrees of influence over the outcome. Influence means that an alternative would have lost without the support of a voter and wins with it.

We want to measure a voter's overall influence.

20.2.1 Banzhaf power index

Definition 20.7 (Banzhaf power index)**.** Given a simple cooperative game (N, v), the *Bhanzhaf power index* of player i is

$$\beta_i \equiv \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S). \tag{70}$$

It is the probability that a player i is *pivotal* in a uniformly random coalition that contains them.

Example 20.8 (The common market)**.** Players are F, G, I, B, N, L with weights 4, 4, 4, 2, 2, 1 and quota 12. We can enumerate a table of 14 winning coalitions. The Banzhaf power indices are $10/32$ for F, G, I; $6/32$ for B, N; and 0 for L.

Theorem 20.9. *Given a weighted voting game and a player $i \in N$, the problem of computing β_i is #P-complete.*

Note. The Banzhaf power index is very easy to approximate using a Monte Carlo algorithm that samples coalitions from the uniform distribution over $2^{N\setminus\{i\}}$ and returns the fraction of coalitions $S : i$ is pivotal in $S \cup \{i\}$.

Theorem 20.10. *Given $\epsilon, \delta > 0$ where ϵ is the desired accuracy and δ is the desired confidence, and $O(\log \frac{1}{\delta} \frac{1}{\epsilon^2})$ samples, the above algorithm returns an estimate $\hat{\beta}_i : |\beta_i - \hat{\beta}_i| < \epsilon$ with probability at least $1 - \delta$.*

Proof. We will use the following lemma.

Lemma 20.11 (Hoeffding)**.** *Let $X_1, \ldots, X_k$ be i.i.id. Bernoulli random variables with $E(X_j) = \mu$. Then*

$$p\left[\left|\frac{1}{k}\sum_{j=1}^{k} X_j - \mu\right| \geq \epsilon\right] \leq 2\exp\left(-2k\epsilon^2\right). \tag{71}$$

For each sample X_j we have that $\mu = \beta_i$.

Plugging in $k = \log \frac{2}{\delta} \frac{1}{2\epsilon^2}$ and $\hat{\beta}_i = \frac{1}{k}\sum_{j=1}^{k} X_j$, we get that the probability that $|\beta_i - \hat{\beta}_i| \geq \epsilon$ is at most $2\exp\left(-\log 2/\delta\right) = \delta$. □

20.3 The influence of voters

The Banzhaf power index quantifies the influence of states but what matters is the influence of voters.

Assume each voter independently votes for each of the two alternatives with probability 0.5. Let α denote the probability that a voter casts a tie-breaking vote in their own state that has population $p = 2k + 1$ and Banzhaf power index β.

The probability that our voter affects the outcome of the election is $\alpha\beta$.

We have estimated β so we need to estimate α. It holds that

$$\alpha = \frac{1}{2^{2k}}\binom{2k}{k} \approx \frac{1}{2^{2k}} \frac{2^{2k}}{\sqrt{\pi k}} = \frac{1}{\sqrt{\pi k}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{p-1}} \tag{72}$$

using Stirling's approximation.

Note. The Electoral College is typically seen as giving an advantage to small states (due to the +2 phenomenon). The irony is that, according to the foregoing analysis, the residents of small states have *significantly less power* than those of large states

21 April 13th, 2022

Today is a class discussion about the material in the second half of the class including fair division, apportionment, and redistricting and allocation.

21.1 Discussion

In knapsack-based participatory budgeting, there is a budget constraint so it is much more likely for larger projects to be funded. In approval-based methods, people do not need to worry about the budget.

People are scared of randomness in Grimmett's randomized apportionment method. In the long-run, having the one seat flip would be beneficial for representation. However, the variance could be very wide in the results of elections and could decrease trust in our institutions.

It is important to note that there is also randomness introduced by noise in Census data via differential privacy.

Given the last election, it is clear that independent redistricting commissions aren't really independent.

It is noted that the geometric target is feasible on *any map* even with conflicting data sets.